A COURSE IN MATHEMATICS

WOODS AND BAILEY

VOLUME II

A COURSE IN
MATHEMATICS

*FOR STUDENTS OF ENGINEERING AND
APPLIED SCIENCE*

BY

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VOLUME II

INTEGRAL CALCULUS
FUNCTIONS OF SEVERAL VARIABLES, SPACE GEOMETRY
DIFFERENTIAL EQUATIONS

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PREFACE

This volume completes the plan of a course in mathematics outlined in the preface to the first volume.

The subject of integration of functions of a single variable is treated in the first eight chapters. Emphasis is here laid upon the fundamental processes, and the conception of the definite integral and its numerous applications are early introduced. Only after the student is well grounded in these matters are the more special methods of evaluating integrals discussed. In this way the student's interest is early aroused in the use of the subject, and he is drilled in those processes which occur in subsequent practice. A new feature of this part of the book is a chapter on simple differential equations in close connection with integration and long before the formal study of differential equations.

With the ninth chapter the study of functions of two or more variables is begun. This is introduced and accompanied by the use of the elements of solid analytic geometry, and the treatment of partial differentiation and of multiple integrals is careful and reasonably complete. A special feature here is the chapter on line integrals. This subject, though generally omitted from elementary texts, is needed by most engineering students in their later work.

The latter part of the work consists of chapters on series, the complex number, and differential equations. In the treatment of differential equations many things properly belonging to an extended treatise on the subject are omitted in order to give the student a concise working knowledge of the types of equations which occur most often in practice.

In conclusion the authors wish to renew their thanks to the members of the mathematical department of the Massachusetts Institute of Technology, and especially to Professor H. W. Tyler, for continued helpful suggestion and criticism, and to extend thanks to their former colleague, Professor W. H. Roever of Washington University, for the construction of the more difficult drawings particularly in space geometry.

Massachusetts Institute of Technology
February, 1909

CONTENTS

CHAPTER I — INFINITESIMALS AND DIFFERENTIALS

ARTICLE PAGE
1–2. Order of infinitesimals 1
3. Fundamental theorems on infinitesimals 4
4. Differentials 6
5–6. Graphical representation 8
7. Formulas for differentials 9
8. Differentials of higher orders 10

CHAPTER II — ELEMENTARY FORMULAS OF INTEGRATION

9. Definition of integration 12
10. Constant of integration 12
11. Fundamental formulas 13
12. Integral of u^n 14
13. Integrals of trigonometric functions 17
14–15. Integrals leading to the inverse trigonometric functions . . 19
16. Integrals of exponential functions 25
17. Collected formulas 26
18. Integration by substitution 27
19. Integration by parts 30
20. Possibility of integration 32
Problems 33

CHAPTER III — DEFINITE INTEGRALS

21. Definition 39
22. Graphical representation 41
23. Generalization 43
24. Properties of definite integrals 45
25. Evaluation of the definite integral by integration 47
26. Change of limits 49
27. Integration by parts 51

ARTICLE PAGE
28. Infinite limits 52
29. Infinite integrand 53
30. The mean value of a function 54
31. Taylor's and Maclaurin's series 55
32–33. Operations with power series 59
Problems 61

CHAPTER IV — APPLICATIONS TO GEOMETRY

34. Element of a definite integral 64
35. Area of a plane curve in Cartesian coördinates 64
36. Area of a plane curve in polar coördinates 67
37. Volume of a solid of revolution 69
38. Volume of a solid with parallel bases 72
39. The prismoidal formula 74
40. Length of a plane curve in rectangular coördinates 75
41. Length of a plane curve in polar coördinates 77
42. The differential of arc 78
43. Area of a surface of revolution 79
Problems 80

CHAPTER V — APPLICATIONS TO MECHANICS

44. Work 86
45. Attraction 86
46. Pressure 88
47. Center of gravity 90
48. Center of gravity of a plane curve 92
49. Center of gravity of a plane area 94
50. Center of gravity of a solid or a surface of revolution of constant density 96
51. Center of pressure 98
Problems 99

CHAPTER VI — INTEGRATION OF RATIONAL FRACTIONS

52. Introduction 103
53–54. Separation into partial fractions 104
55–57. Proof of the possibility of separation into partial fractions . 109
58. Integration of rational fractions 113
Problems 117

CHAPTER VII — SPECIAL METHODS OF INTEGRATION

ARTICLE PAGE

59. Rationalization . . . 119
60. Integrand containing fractional powers of $a + bx$ . . . 119
61. Integrand containing fractional powers of $a + bx^n$ . . . 120
62. Integrand containing integral powers of $\sqrt{a + bx + x^2}$ . . . 121
63. Integrand containing integral powers of $\sqrt{a + bx - x^2}$ . . . 122
64. Integration of trigonometric functions . . . 123
65. Integrals of the forms $\int \sin^n x\,dx$ and $\int \cos^n x\,dx$ . . . 123
66. Integrals of the form $\int \sin^m x \cos^n x\,dx$ . . . 124
67. Integrals of the forms $\int \tan^n x\,dx$ and $\int \text{ctn}^n x\,dx$ . . . 126
68. Integrals of the forms $\int \sec^n x\,dx$ and $\int \csc^n x\,dx$ . . . 127
69. Integrals of the forms $\int \tan^m x \sec^n x\,dx$ and $\int \text{ctn}^m x \csc^n x\,dx$. 128
70. The substitution $\tan\frac{x}{2} = z$ . . . 129
71. Algebraic reduction formulas . . . 130
72–73. Proof of reduction formulas . . . 131
74. Trigonometric reduction formulas . . . 134
75. Use of tables of integrals . . . 136
Problems . . . 137

CHAPTER VIII — INTEGRATION OF SIMPLE DIFFERENTIAL EQUATIONS

76. Definitions . . . 141
77. The equation $Mdx + Ndy = 0$ when the variables can be separated . . . 144
78–79. The homogeneous equation $Mdx + Ndy = 0$ . . . 145
80. The linear equation of the first order . . . 146
81. Bernouilli's equation . . . 148
82. Certain equations of the second order . . . 149
Problems . . . 154

CHAPTER IX — FUNCTIONS OF SEVERAL VARIABLES

83. Functions of more than one variable . . . 158
84. Rectangular coördinates in space . . . 158
85–86. Graphical representation of a function of two variables . . 159
87. Surfaces of revolution . . . 168

ARTICLE PAGE
88. Cylinders 169
89–90. Space curves 170
91. Ruled surfaces 172
Problems 174

CHAPTER X — PLANE AND STRAIGHT LINE

92. Projection 176
93. Distance between two points 178
94. Length of a space curve in rectangular coördinates . . . 179
95–96. Direction of a straight line 180
97. Direction of a space curve 182
98. Angle between two straight lines 183
99. Direction of the normal to a plane 184
100. Normal equation of a plane 185
101. Angle between two planes 186
102. Determination of the direction cosines of any straight line . 186
103–104. Equations of a straight line determined by two points . . 187
105. Equations of a tangent line to a space curve 189
106. Equations of a straight line in terms of its direction cosines and a known point upon it 190
107. Problems on the plane and the straight line 190
108. Change of coördinates 193
Problems 195

CHAPTER XI — PARTIAL DIFFERENTIATION

109. Partial derivatives 198
110. Increment and total differential of a function of several variables 200
111. Derivative of $f(x, y)$ when x and y are functions of t . . 202
112. Tangent plane to a surface 204
113. Derivatives of $f(x, y)$ when x and y are functions of s and t. 206
114. Property of the total differential 209
115. Implicit functions 210
116–117. Higher partial derivatives 212
118. Higher derivatives of $f(x, y)$ when x and y are functions of t, or of s and t 214
119. Differentiation of a definite integral 216
Problems 217

CHAPTER XII — MULTIPLE INTEGRALS

ARTICLE PAGE

120. Double integral with constant limits 222
121. Graphical representation 225
122–123. Double integral with variable limits 226
124. Computation of a double integral 228
125. Double integral in polar coördinates 229
126. Triple integrals 230
127. Cylindrical and polar coördinates 231
128. Elements of volume in cylindrical and in polar coördinates . 232
129. Change of coördinates 234
Problems 234

CHAPTER XIII — APPLICATIONS OF MULTIPLE INTEGRALS

130–132. Moment of inertia of a plane area 236
133–134. Area bounded by a plane curve 239
135. Area of any surface 241
136. Center of gravity 245
137–138. Center of gravity of a plane area 246
139. Center of gravity of a solid 248
140. Volume 249
141. Moment of inertia of a solid 251
142. Attraction 252
Problems 253

CHAPTER XIV — LINE INTEGRALS AND EXACT DIFFERENTIALS

143. Definition 258
144. Fundamental theorem 261
145. Line integrals of the first kind, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ 263
146. Line integrals of the second kind, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ 267
147. Exact differentials 269
148–149. The integrating factor 271
150–151. Stokes's theorem 274
Problems 276

CHAPTER XV — INFINITE SERIES

152. Convergence 279
153. Comparison test for convergence 280
154. The ratio test for convergence 281

ARTICLE PAGE
155. Absolute convergence 283
156. The power series 284
157. Maclaurin's and Taylor's series 287
158. Taylor's series for functions of several variables 288
159–160. Fourier's series 290
161–162. The indeterminate form $\frac{0}{0}$ 295
163. The indeterminate form $\frac{\infty}{\infty}$ 298
164. Other indeterminate forms 300
Problems 301

CHAPTER XVI — COMPLEX NUMBERS

165. Graphical representation 304
166. Addition and subtraction 305
167. Multiplication and division 306
168. Involution and evolution 306
169. Exponential and trigonometric functions 307
170. The logarithmic function 310
171. Functions of a complex variable in general 311
172. Conjugate functions 313
Problems 314

CHAPTER XVII — DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

173. Introduction 316
174. Solution by series 318
175. Equations not of the first degree in the derivative 318
176. Equations solvable for p 319
177. Equations solvable for y 320
178. Equations solvable for x 321
179–181. Envelopes 322
182. Singular solutions 325
183. Orthogonal trajectories 327
184. Differential equation of the first order in three variables. The integrable case 328
185. Two differential equations of the first order in three variables 332
186. Differential equation of the first order in three variables. The nonintegrable case 335
Problems 336

CHAPTER XVIII — THE LINEAR DIFFERENTIAL EQUATION

ARTICLE PAGE
187. Definitions 340
188. The equation of the first order with constant coefficients . 341
189. The operator $\frac{1}{D-a}$ 342
190. The equation of the second order with constant coefficients . 343
191. Solution by partial fractions 347
192. The general equation with constant coefficients 348
193. Solution by undetermined coefficients 350
194. Systems of linear differential equations with constant coefficients 353
195. The linear differential equation with variable coefficients . 354
196. Solution by series 356
Problems 360

CHAPTER XIX — PARTIAL DIFFERENTIAL EQUATIONS

197. Introduction 363
198. Special forms of partial differential equations 363
199. The linear partial differential equation of the first order . 364
200. Laplace's equation in the plane 366
201. Laplace's equation in three dimensions 368
Problems 370

ANSWERS 373
INDEX 399

A COURSE IN MATHEMATICS

CHAPTER I

INFINITESIMALS AND DIFFERENTIALS

1. Order of infinitesimals. *An infinitesimal is a variable which approaches zero as a limit.* The word "infinitesimal" in the mathematical sense must not be considered as meaning "very minute." For example, the size of an atom of matter is not a mathematical infinitesimal, since that size is regarded as perfectly definite. That a quantity should be infinitesimal it is essential that it can be made smaller than any assigned quantity. In operating with infinitesimals it is not, however, necessary to think of them as microscopic or of negligible size. They are finite quantities and obey all the laws of multiplication, division, etc., like finite quantities. It is generally the last step in a problem involving infinitesimals to determine the limit of some expression, usually a quotient or a sum, when the infinitesimals contained in it approach zero.

Ex. 1. To find the velocity of a moving body (I, § 106)* it is necessary to find the limit of the quotient $\frac{\Delta s}{\Delta t}$, as Δs and Δt approach zero as a limit, where Δs is the space traversed in the time Δt. Here Δs and Δt are infinitesimals and the velocity is the limit of the quotient of two infinitesimals.

Ex. 2. To find the area of a circle it is customary to inscribe in the circle a regular polygon of n sides and to divide the polygon into n triangles by radii of the circle drawn to the vertices of the polygon. When n is increased without limit, the area of each triangle is infinitesimal, and the area of the circle is the limit of the sum of the infinitesimal areas of an indefinitely great number of triangles.

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* References preceded by I refer to Vol. I.

In a mathematical discussion involving infinitesimals there will usually be two or more infinitesimals so related that as one approaches zero the others do also. Any two of these may be compared by determining the limit of their ratio. We have accordingly the following definitions:

Two infinitesimals are of the same order when the limit of their ratio is a finite quantity not zero.

An infinitesimal β is of higher order than an infinitesimal α if the limit of the ratio $\frac{\beta}{\alpha}$ is zero.

Ex. 3. Let $\beta = \sin\alpha$ and $\gamma = 1 - \cos\alpha$, where α is an infinitesimal angle.

Then
$$\text{Lim}\,\frac{\beta}{\alpha} = \text{Lim}\,\frac{\sin\alpha}{\alpha} = 1,$$
$$\text{Lim}\,\frac{\gamma}{\alpha} = \text{Lim}\,\frac{1-\cos\alpha}{\alpha} = 0. \qquad \text{(I, § 151)}$$

Hence β is of the same order as α, and γ is of higher order.

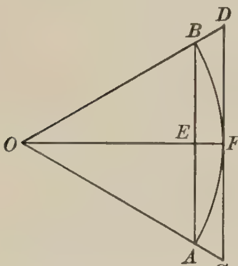

Fig. 1

Ex. 4. Let the arc AB (fig. 1) be an arc of a circle of radius a with center at O, the chord AB the side of an inscribed regular polygon of n sides, and CD the side of a regular circumscribed polygon. Also let

$\alpha =$ the area of the triangle AOB,
$\beta =$ the area of the triangle COD,
$\gamma =$ the area of the trapezoid $ABDC$.

Then if n is indefinitely increased, α, β, and γ are infinitesimal. To compute their values, draw OE perpendicular to AB and CD. Then

$$\angle AOB = \frac{2\pi}{n},\ \angle EOB = \frac{\pi}{n},\ OE = a\cos\frac{\pi}{n},\ EB = a\sin\frac{\pi}{n},\ FD = a\tan\frac{\pi}{n}.$$

From this it follows that

$$\alpha = OE\cdot\frac{AB}{2} = OE\cdot EB = a^2\sin\frac{\pi}{n}\cos\frac{\pi}{n},$$
$$\beta = OF\cdot\frac{CD}{2} = OF\cdot FD = a^2\tan\frac{\pi}{n},$$
$$\gamma = \frac{AB+CD}{2}\cdot EF = (EB+FD)(OF-OE)$$
$$= a^2\left(\sin\frac{\pi}{n}+\tan\frac{\pi}{n}\right)\left(1-\cos\frac{\pi}{n}\right) = a^2\,\frac{\sin^3\frac{\pi}{n}}{\cos\frac{\pi}{n}}.$$

Hence
$$\operatorname{Lim}\frac{\beta}{\alpha}=\operatorname{Lim}\frac{1}{\cos^2\frac{\pi}{n}}=1,$$

$$\operatorname{Lim}\frac{\gamma}{\alpha}=\operatorname{Lim}\frac{\sin^2\frac{\pi}{n}}{\cos^2\frac{\pi}{n}}=0.$$

Therefore β is of the same order as α, and γ is of higher order.

2. Particular importance attaches to the case in which the limit of the ratio of two infinitesimals is unity. Suppose, for example, that

$$\operatorname{Lim}\frac{\beta}{\alpha}=1.$$

Then, by the definition of a limit (I, § 53),

$$\frac{\beta}{\alpha}=1+\epsilon,$$

where ϵ approaches zero as α approaches zero. Hence

$$\beta=\alpha+\alpha\epsilon.$$

Now the term $\alpha\epsilon$ is an infinitesimal of higher order than α, for $\operatorname{Lim}\frac{\alpha\epsilon}{\alpha}=\operatorname{Lim}\epsilon=0$, so that β and α differ by an infinitesimal of higher order than each of them.

Conversely, let β and α differ by an infinitesimal of higher order than either of them, i.e. let

$$\beta=\alpha+\gamma,$$

where, by hypothesis, $\operatorname{Lim}\frac{\gamma}{\alpha}=0$. Then

$$\operatorname{Lim}\frac{\beta}{\alpha}=\operatorname{Lim}\left(1+\frac{\gamma}{\alpha}\right)=1.$$

We have accordingly proved that the two statements, "Two infinitesimals β and α differ by an infinitesimal of higher order" and "$\operatorname{Lim}\frac{\beta}{\alpha}=1$" are equivalent.

Ex. 1. The infinitesimals α and $\sin\alpha$ differ by an infinitesimal of higher order (I, § 151).

Ex. 2. The areas of the triangles AOB and COD (Ex. 4, § 1) differ by an infinitesimal of higher order, namely, the area of the trapezoid $ABDC$.

3. Fundamental theorems on infinitesimals. There are two important problems which arise in the use of infinitesimals, namely:

1. A *quotient problem*: to find the limit of the quotient of two infinitesimals as each approaches zero.

2. A *sum problem*: to find the limit of the sum of a number of infinitesimals as the number increases without limit and each infinitesimal approaches zero.

Each of these problems has been illustrated in § 1; for each there is a fundamental theorem as follows:

1. *If the quotient of two infinitesimals has a limit, that limit is unaltered by replacing each infinitesimal by another which differs from it by an infinitesimal of higher order.*

2. *If the sum of n positive infinitesimals has a limit, as n increases indefinitely, that limit is unaltered by replacing each infinitesimal by another which differs from it by an infinitesimal of higher order.*

To prove theorem 1, let α and β be two infinitesimals and let α_1 and β_1 be two others which differ from α and β respectively by infinitesimals of higher order. Then we have (§ 2)

$$\text{Lim}\,\frac{\alpha}{\alpha_1}=1, \qquad \text{Lim}\,\frac{\beta}{\beta_1}=1,$$

whence

$$\alpha=\alpha_1+\epsilon_1\alpha_1, \qquad \beta=\beta_1+\epsilon_2\beta_1,$$

where ϵ_1 and ϵ_2 approach zero as α and β approach zero. Then

$$\frac{\beta}{\alpha}=\frac{\beta_1+\epsilon_2\beta_1}{\alpha_1+\epsilon_1\alpha_1}=\frac{\beta_1}{\alpha_1}\cdot\frac{1+\epsilon_2}{1+\epsilon_1}.$$

Therefore

$$\text{Lim}\,\frac{\beta}{\alpha}=\text{Lim}\left(\frac{\beta_1}{\alpha_1}\cdot\frac{1+\epsilon_2}{1+\epsilon_1}\right)=\text{Lim}\,\frac{\beta_1}{\alpha_1}\,\text{Lim}\,\frac{1+\epsilon_2}{1+\epsilon_1}=\text{Lim}\,\frac{\beta_1}{\alpha_1}.$$

Ex. 1. Since the sine of an angle differs from the angle by an infinitesimal of higher order,

$$\text{Lim}\,\frac{\sin 3\alpha}{\sin 2\alpha}=\text{Lim}\,\frac{3\alpha}{2\alpha}=\frac{3}{2}.$$

To show this directly, we may write

$$\frac{\sin 3\alpha}{\sin 2\alpha}=\frac{3\sin\alpha-4\sin^3\alpha}{2\sin\alpha\cos\alpha}=\frac{3}{2\cos\alpha}-\frac{2\sin^2\alpha}{\cos\alpha}.$$

Therefore

$$\text{Lim}\,\frac{\sin 3\alpha}{\sin 2\alpha}=\frac{3}{2}.$$

To prove theorem 2, let $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n$ be n positive infinitesimals so related that, as n increases indefinitely, each of the infinitesimals approaches zero and their sum approaches a limit; and let $\beta_1, \beta_2, \beta_3, \cdots, \beta_n$ be n other infinitesimals such that

$$\operatorname{Lim} \frac{\beta_1}{\alpha_1} = 1,\ \operatorname{Lim} \frac{\beta_2}{\alpha_2} = 1,\ \operatorname{Lim} \frac{\beta_3}{\alpha_3} = 1,\ \cdots,\ \operatorname{Lim} \frac{\beta_n}{\alpha_n} = 1.$$

We wish to show that

$$\operatorname*{Lim}_{n=\infty} (\beta_1 + \beta_2 + \beta_3 + \cdots + \beta_n) = \operatorname*{Lim}_{n=\infty} (\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n).$$

Now (§ 2)

$$\begin{aligned} \beta_1 &= \alpha_1 + \epsilon_1 \alpha_1, \\ \beta_2 &= \alpha_2 + \epsilon_2 \alpha_2, \\ \beta_3 &= \alpha_3 + \epsilon_3 \alpha_3, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \beta_n &= \alpha_n + \epsilon_n \alpha_n; \end{aligned}$$

whence

$$\begin{aligned} \operatorname*{Lim}_{n=\infty} (\beta_1 + \beta_2 + \beta_3 + \cdots + \beta_n) &= \operatorname*{Lim}_{n=\infty} (\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n) \\ &\quad + \operatorname*{Lim}_{n=\infty} (\epsilon_1 \alpha_1 + \epsilon_2 \alpha_2 + \epsilon_3 \alpha_3 + \cdots + \epsilon_n \alpha_n). \end{aligned}$$

Now let γ be a positive quantity which is equal to the largest numerical value of the quantities $\epsilon_1, \epsilon_2, \epsilon_3, \cdots, \epsilon_n$.

Then

$$-\gamma \leqq \epsilon_1 \leqq \gamma,$$

whence

$$-\gamma \alpha_1 \leqq \epsilon_1 \alpha_1 \leqq \gamma \alpha_1,$$

since by hypothesis α_1 is positive and may multiply the inequality without change of sign. Similarly,

$$\begin{aligned} -\gamma \alpha_2 &\leqq \epsilon_2 \alpha_2 \leqq \gamma \alpha_2, \\ -\gamma \alpha_3 &\leqq \epsilon_3 \alpha_3 \leqq \gamma \alpha_3, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ -\gamma \alpha_n &\leqq \epsilon_n \alpha_n \leqq \gamma \alpha_n; \end{aligned}$$

whence

$$\begin{aligned} -\gamma (\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n) &\leqq \epsilon_1 \alpha_1 + \epsilon_2 \alpha_2 + \epsilon_3 \alpha_3 + \cdots + \epsilon_n \alpha_n \\ &\leqq \gamma (\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n). \end{aligned}$$

As n increases indefinitely, $\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n$ approaches a finite limit by hypothesis, and γ approaches zero.

Therefore $\operatorname*{Lim}_{n=\infty} (\epsilon_1\alpha_1 + \epsilon_2\alpha_2 + \epsilon_3\alpha_3 + \cdots + \epsilon_n\alpha_n) = 0$, and hence

$$\operatorname*{Lim}_{n=\infty} (\beta_1 + \beta_2 + \beta_3 + \cdots + \beta_n) = \operatorname*{Lim}_{n=\infty} (\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n).$$

The theorem is thus proved for positive infinitesimals.

Ex. 2. The limit of the sum of the areas of n triangles such as AOB (fig. 1) is the same as the limit of the sum of the areas of n triangles such as COD, when n is indefinitely increased.

The theorem is also true if the infinitesimals are all negative, since to change the sign of each infinitesimal is simply to change the sign of the limit of the sum. If the infinitesimals are not all of the same sign, however, the theorem is not necessarily true.

Ex. 3. Let

$$\alpha_1 = \frac{1}{\sqrt{n}},\quad \alpha_2 = -\frac{1}{\sqrt{n}},\quad \alpha_3 = \frac{1}{\sqrt{n}},\quad \alpha_4 = -\frac{1}{\sqrt{n}},\ \text{etc.},$$

$$\beta_1 = \alpha_1 + \frac{1}{n},\quad \beta_2 = \alpha_2 + \frac{1}{n},\quad \beta_3 = \alpha_3 + \frac{1}{n},\quad \beta_4 = \alpha_4 + \frac{1}{n},\ \text{etc.}$$

Then
$$\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n = 0,\ \text{or } = \frac{1}{\sqrt{n}},$$
$$\beta_1 + \beta_2 + \beta_3 + \cdots + \beta_n = 1,\ \text{or } = 1 + \frac{1}{\sqrt{n}},$$
according as n is even or odd; and

$$\operatorname*{Lim}_{n=\infty} (\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n) = 0,$$
$$\operatorname*{Lim}_{n=\infty} (\beta_1 + \beta_2 + \beta_3 + \cdots + \beta_n) = 1.$$

4. Differentials. The process of differentiation is an illustration of the quotient problem of § 3. For if $y = f(x)$ is a continuous function of x which has the derivative $f'(x)$, and Δx and Δy are the corresponding infinitesimal increments of x and y, then, by definition,

$$\operatorname*{Lim}_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = f'(x). \tag{1}$$

It appears from (1) that Δx and Δy are infinitesimals of the same order, except in the cases in which $f'(x)$ is 0 or ∞. Moreover, (1) may be written

$$\frac{\Delta y}{\Delta x} = f'(x) + \epsilon,$$

where $\lim\limits_{\Delta x \doteq 0} \epsilon = 0$, and hence

$$\Delta y = f'(x)\Delta x + \epsilon\Delta x. \tag{2}$$

It appears, then, that $f'(x)\Delta x$ differs from Δy by an infinitesimal of higher order than Δx, and $f'(x)\Delta x$ may therefore be used in place of Δy in problems involving limits of quotients and sums. The quantity $f'(x)\Delta x$ is called the *differential* of y, and is represented by the symbol dy. Accordingly

$$dy = f'(x)\Delta x. \tag{3}$$

Now in the special case in which $y = x$, formula (3) reduces to

$$dx = \Delta x. \tag{4}$$

Hence we may write (3) as

$$dy = f'(x)\,dx. \tag{5}$$

To sum this up: *The differential of the independent variable is equal to the increment of the variable; the differential of the function is equal to the differential of the independent variable multiplied by the derivative of the function, and differs from the increment of the function by an infinitesimal of higher order.*

From this point of view the derivative is called the *differential coefficient.*

The use of differentials instead of increments is justified by the fundamental theorems of § 3 in many problems which are eventually to involve the limit of a quotient or the limit of a sum.

It is to be emphasized that dx and dy are finite quantities, subject to all the laws governing such quantities, and are not to be thought of as exceedingly minute. Consequently both sides of (5) may be divided by dx, with the result

$$f'(x) = \frac{dy}{dx}.$$

That is, the derivative is the quotient of two differentials. This explains the notation already chosen for the derivative.

Ex. 1. Let $y = x^3$.

We may increase x by an increment Δx equal to dx. Then

$$\Delta y = (x + dx)^3 - x^3 = 3\,x^2dx + 3\,x\,(dx)^2 + (dx)^3.$$

On the other hand, by definition,

$$dy = 3\,x^2dx.$$

It appears that Δy and dy differ by the expression $3\,x\,(dx)^2 + (dx)^3$, which is an infinitesimal of higher order than dx.

Ex. 2. If a volume v of a perfect gas at a constant temperature is under the pressure p, then $v = \frac{k}{p}$, where k is a constant. Now let the pressure be increased by an amount $\Delta p = dp$. The actual change in the volume of the gas is then the increment

$$\Delta v = \frac{k}{p + dp} - \frac{k}{p} = -\frac{kdp}{p\,(p + dp)} = -\frac{kdp}{p^2}\left(\frac{1}{1 + \frac{dp}{p}}\right).$$

The differential of v is, however,

$$dv = -\frac{kdp}{p^2},$$

which differs from Δv by an infinitesimal of higher order. The differential dv may, accordingly, be used in place of Δv in problems which involve the limit of quotients or sums of this and other infinitesimals.

5. Graphical representation. The distinction between the increment and the differential may be illustrated graphically as follows:

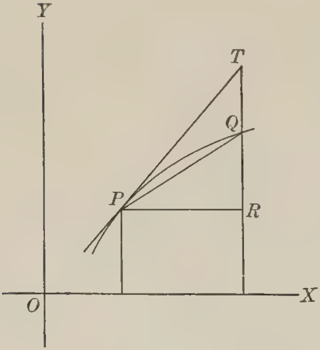

FIG. 2

Let the function be represented by the curve $y = f(x)$ (fig. 2). Let $P(x, y)$ be any point of the curve, and $Q(x + \Delta x, y + \Delta y)$ a neighboring point. Draw the lines PR and RQ parallel to the axes, the chord PQ, and the tangent PT. Then

$$\Delta x = dx = PR, \quad \Delta y = RQ, \quad f'(x) = \tan RPT,$$

and $$dy = f'(x)\,dx = (\tan RPT)\,PR = RT.$$

Hence the increment and the differential differ by QT. That QT is an infinitesimal of higher order than PR may be shown as follows:

$$\text{Lim}\,\frac{QT}{PR}=\text{Lim}\,\frac{RT-RQ}{PR}=\frac{RT}{PR}-\text{Lim}\,\frac{RQ}{PR}$$
$$=\tan RPT-\text{Lim}\,(\tan RPQ)=0.$$

6. The formula

$$dy=f'(x)\,dx \tag{1}$$

has been obtained on the hypothesis that x is the independent variable and $y=f(x)$.

Consider now the case $y=f(x)$, where $x=\phi(t)$ and t is the independent variable. By substitution we have

$$y=f(x)=f[\phi(t)]=F(t).$$

Then, by (1), $$dy=F'(t)\,dt. \tag{2}$$

But by I, § 96, (7), $$F'(t)=f'(x)\cdot\phi'(t).$$

Substituting in (2), we have

$$dy=f'(x)\cdot\phi'(t)\,dt. \tag{3}$$

But since t is the independent variable and $x=\phi(t)$, we have, from (1),

$$dx=\phi'(t)\,dt.$$

Substituting in (3), we have

$$dy=f'(x)\,dx.$$

This is the same form as (1). Therefore (1) is always true whether x is the independent variable or not.

7. Formulas for differentials. By virtue of the results of the preceding article, the formulas for differentials can be derived from the formulas for the corresponding derivatives.

For example, since the derivative of u^n with respect to u is nu^{n-1},

$$d(u^n)=nu^{n-1}du,$$

whether u be an independent variable or the function of another variable.

Proceeding as in the above example, we derive the following formulas:

$$d(u+c)=du, \tag{1}$$

$$d(cu)=c\,du, \tag{2}$$

$$d(u+v)=du+dv, \tag{3}$$

$$d(uv)=u\,dv+v\,du, \tag{4}$$

$$d\left(\frac{u}{v}\right)=\frac{v\,du-u\,dv}{v^2}, \tag{5}$$

$$d(u^n)=nu^{n-1}du, \tag{6}$$

$$d\sin u=\cos u\,du, \tag{7}$$

$$d\cos u=-\sin u\,du, \tag{8}$$

$$d\tan u=\sec^2 u\,du, \tag{9}$$

$$d\operatorname{ctn} u=-\csc^2 u\,du, \tag{10}$$

$$d\sec u=\sec u\tan u\,du, \tag{11}$$

$$d\csc u=-\csc u\operatorname{ctn} u\,du, \tag{12}$$

$$d\sin^{-1}u=\pm\frac{du}{\sqrt{1-u^2}}, \tag{13}$$

$$d\cos^{-1}u=\mp\frac{du}{\sqrt{1-u^2}}, \tag{14}$$

$$d\tan^{-1}u=\frac{du}{1+u^2}, \tag{15}$$

$$d\operatorname{ctn}^{-1}u=-\frac{du}{1+u^2}, \tag{16}$$

$$d\sec^{-1}u=\pm\frac{du}{u\sqrt{u^2-1}}, \tag{17}$$

$$d\csc^{-1}u=\mp\frac{du}{u\sqrt{u^2-1}}, \tag{18}$$

$$d\,e^u=e^u\,du, \tag{19}$$

$$d\log u=\frac{du}{u}, \tag{20}$$

$$d\,a^u=a^u\log a\,du, \tag{21}$$

$$d\log_a u=\log_a e\,\frac{du}{u}. \tag{22}$$

8. Differentials of higher orders. By definition

$$dy=f'(x)\,dx. \tag{1}$$

But dy is itself a function of x, and may, accordingly, have a differential, which will be denoted by d^2y; i.e. $d(dy)=d^2y$, and, in general, $d(d^{n-1}y)=d^ny$. The differentials $d^2y, d^3y, \cdots, d^ny$ are called differentials of the second, the third, $\cdots$, the nth order respectively.

When x is the independent variable, its increment dx may be taken as independent of x. With this assumption, applying our definition of a differential to (1), we have

$$d^2y=d[f'(x)\,dx]=[f''(x)\,dx]\,dx=f''(x)\,dx^2,$$

where dx^2 is written for convenience instead of $(dx)^2$.

In like manner,

$$d^3y=f'''(x)\,dx^3,$$

and, in general,

$$d^ny=f^{(n)}(x)\,dx^n.$$

The reason for the notation used for derivatives of the second, the third, · · ·, the nth orders is now apparent.

If x is not the independent variable, the expressions for the differentials become more complex, since dx cannot be assumed as independent of x. We have, then,

$$d^2y = d\,[f'(x)\,dx] = f''(x)\,dx^2 + f'(x)\,d^2x, \quad \text{(by (4), § 7)}$$

where, if $x = \phi(t)$, $dx = \phi'(t)\,dt$, and $d^2x = \phi''(t)\,dt^2$, t being the independent variable. It appears, then, that the formula for the second differential is not the same when written in terms of the independent variable as it is when written in terms of a function of that variable. This is true of all differentials except the first (see § 6).

Hence the higher differentials are not as convenient as the first, and the student is advised to avoid their use at present.

CHAPTER II

ELEMENTARY FORMULAS OF INTEGRATION

9. Definition of integration. *The process of finding a function when its differential is known is called integration.* This is evidently the same as the process of finding a function when its derivative is known, called integration in I, § 110.

In that place integration was performed by rewriting the derivative in such a manner that we could recognize, by the formulas of differentiation, the function of which it is the derivative. But this method can be applied only in the simpler cases. For the more complex cases it is necessary to have *formulas of integration*, which can evidently be derived from the formulas for differentials (§ 7).

Using the symbol $\int$ to denote integration, it is evident from the definition that if

$$f(x)\,dx = dF(x),$$

then

$$\int f(x)\,dx = F(x).$$

The expression $f(x)\,dx$ is said to be *under* the sign of integration, and $f(x)$ is called the *integrand*. $F(x)$ is called the *integral* of $f(x)\,dx$.

10. Constant of integration. Two functions which differ only by a constant have the same derivative and hence the same differential; and conversely, if two functions have the same differential, they differ only by a constant (I, § 110; II, § 30).

Hence if

$$f(x)\,dx = dF(x), \tag{1}$$

it follows that

$$f(x)\,dx = d[F(x) + C], \tag{2}$$

where C is any constant.

Rewriting (1) and (2) as formulas of integration, we have

$$\int f(x)\,dx = F(x) \tag{3}$$

and

$$\int f(x)\,dx = F(x) + C. \tag{4}$$

It is evident that (3) is but a special case of (4), and hence that all integrals ought to be written in the latter form. The constant C is called the *constant of integration* and is independent of the form of the integrand. For the sake of brevity it will be omitted from the formulas of integration, but must be added in all integrals evaluated by means of them. As noted in I, § 110, its value is determined by the special conditions of the problem in which the integral occurs.

11. Fundamental formulas. The two formulas

$$\int c\,du = c\int du \tag{1}$$

and
$$\int (du + dv + dw + \cdots) = \int du + \int dv + \int dw + \cdots \tag{2}$$

are of fundamental importance, one or both of them being used in the course of almost every integration. Stated in words, they are as follows:

(1) *A constant factor may be changed from one side of the sign of integration to the other.*

(2) *The integral of a sum of a finite number of functions is the sum of the integrals of the separate functions.*

To prove (1), we note that since $c\,du = d(cu)$, it follows that

$$\int c\,du = \int d(cu) = cu = c\int du.$$

In like manner, to prove (2), since

$$du + dv + dw + \cdots = d(u + v + w + \cdots),$$

we have

$$\begin{aligned}\int (du + dv + dw + \cdots) &= \int d(u + v + w + \cdots)\\ &= u + v + w + \cdots\\ &= \int du + \int dv + \int dw + \cdots.\end{aligned}$$

The application of these formulas is illustrated in the following articles.

12. Integral of u^n. Since

$$d(u^m) = mu^{m-1}du,$$

it follows that

$$\int mu^{m-1}du = u^m.$$

But by § 11 (1), if $m \neq 0$,

$$m\int u^{m-1}du = \int mu^{m-1}du = u^m,$$

or

$$\int u^{m-1}du = \frac{u^m}{m}\cdot$$

Placing $m = n+1$, we have the formula

$$\int u^n du = \frac{u^{n+1}}{n+1} \tag{1}$$

for all values of n except $n = -1$.

If $n = -1$, the differential under the integral sign in (1) becomes $\frac{du}{u}$, which is recognized as $d(\log u)$.

Therefore

$$\int \frac{du}{u} = \log u. \tag{2}$$

In applying these formulas, the problem is to choose for u some function of x which will bring the given integral, if possible, under one of the formulas. The form of the integrand often suggests the function of x which should be chosen for u.

Ex. 1. Find the value of $\int\left(ax^2 + bx + \frac{c}{x} + \frac{e}{x^2}\right)dx.$

Applying § 11 (2) and then § 11 (1), we have

$$\begin{aligned}\int\left(ax^2 + bx + \frac{c}{x} + \frac{e}{x^2}\right)dx &= \int ax^2 dx + \int bx\,dx + \int \frac{c}{x}dx + \int \frac{e}{x^2}dx \\ &= a\int x^2 dx + b\int x\,dx + c\int\frac{dx}{x} + e\int x^{-2}dx.\end{aligned}$$

The first, the second, and the fourth of these integrals may be evaluated by formula (1), and the third by formula (2), where $u = x$, the results being respectively $\frac{1}{3}ax^3$, $\frac{1}{2}bx^2$, $-\frac{e}{x}$, and $c\log x$.

Therefore

$$\int\left(ax^2 + bx + \frac{c}{x} + \frac{e}{x^2}\right)dx = \frac{1}{3}ax^3 + \frac{1}{2}bx^2 + c\log x - \frac{e}{x} + C.$$

Ex. 2. Find the value of $\int (x^2 + 2)\, x\, dx$.

If the factors of the integrand are multiplied together, we have

$$\int (x^2 + 2)\, x\, dx = \int (x^3 + 2x)\, dx,$$

which may be evaluated by the same method as that used in Ex. 1, the result being $\frac{1}{4} x^4 + x^2 + C$.

Or, we may let $x^2 + 2 = u$, whence $2x\, dx = du$, so that $x\, dx = \frac{1}{2} du$. Hence

$$\begin{aligned} \int (x^2 + 2)\, x\, dx &= \int \tfrac{1}{2} u\, du = \tfrac{1}{2} \int u\, du \\ &= \frac{1}{2} \cdot \frac{u^2}{2} + C \\ &= \tfrac{1}{4} (x^2 + 2)^2 + C. \end{aligned}$$

Instead of actually writing out the integral in terms of u, we may note that $x\, dx = \frac{1}{2} d(x^2 + 2)$, and proceed as follows:

$$\begin{aligned} \int (x^2 + 2)\, x\, dx &= \int (x^2 + 2) \tfrac{1}{2} d(x^2 + 2) \\ &= \tfrac{1}{2} \int (x^2 + 2)\, d(x^2 + 2) \\ &= \tfrac{1}{4} (x^2 + 2)^2 + C. \end{aligned}$$

Comparing the two values of the integral found by the two methods of integration, we see that they differ only by the constant unity, which may be made a part of the constant of integration.

Ex. 3. Find the value of $\int (ax^2 + 2bx)^3 (ax + b)\, dx$.

Let $ax^2 + 2bx = u$. Then $(2ax + 2b)\, dx = du$, so that $(ax + b)\, dx = \frac{1}{2} du$. Hence

$$\begin{aligned} \int (ax^2 + 2bx)^3 (ax + b)\, dx &= \int \tfrac{1}{2} u^3\, du \\ &= \frac{1}{2} \int u^3\, du = \frac{1}{2} \cdot \frac{u^4}{4} + C \\ &= \tfrac{1}{8} (ax^2 + 2bx)^4 + C. \end{aligned}$$

Or, the last part of the work may be arranged as follows:

$$\begin{aligned} \int (ax^2 + 2bx)^3 (ax + b)\, dx &= \int (ax^2 + 2bx)^3 \tfrac{1}{2} d(ax^2 + 2bx) \\ &= \tfrac{1}{2} \int (ax^2 + 2bx)^3 d(ax^2 + 2bx) \\ &= \tfrac{1}{8} (ax^2 + 2bx)^4 + C. \end{aligned}$$

Ex. 4. Find the value of $\int \frac{4(ax+b)\,dx}{ax^2+2bx}$.

As in Ex. 3, let $ax^2+2bx=u$. Then $(2ax+2b)dx=du$, so that $(ax+b)dx=\frac{1}{2}du$. Hence

$$\begin{aligned}\int \frac{4(ax+b)\,dx}{ax^2+2bx} &= \int \frac{2\,du}{u} = 2\int \frac{du}{u}\\ &= 2\log u + C\\ &= 2\log(ax^2+2bx) + C\\ &= \log(ax^2+2bx)^2 + C.\end{aligned}$$

Or,

$$\begin{aligned}\int \frac{4(ax+b)\,dx}{ax^2+2bx} &= \int \frac{2\,d(ax^2+2bx)}{ax^2+2bx}\\ &= 2\int \frac{d(ax^2+2bx)}{ax^2+2bx}\\ &= 2\log(ax^2+2bx) + C\\ &= \log(ax^2+2bx)^2 + C.\end{aligned}$$

Ex. 5. Find the value of $\int (e^{ax}+b)^2 e^{ax}\,dx$.

Let $e^{ax}+b=u$. Then $e^{ax}a\,dx=du$. Hence

$$\begin{aligned}\int (e^{ax}+b)^2 e^{ax}\,dx &= \int u^2\frac{du}{a}\\ &= \frac{1}{a}\int u^2\,du\\ &= \frac{1}{3a}u^3 + C\\ &= \frac{1}{3a}(e^{ax}+b)^3 + C.\end{aligned}$$

Or,

$$\begin{aligned}\int (e^{ax}+b)^2 e^{ax}\,dx &= \int \frac{1}{a}(e^{ax}+b)^2\,d(e^{ax}+b)\\ &= \frac{1}{a}\int (e^{ax}+b)^2\,d(e^{ax}+b)\\ &= \frac{1}{3a}(e^{ax}+b)^3 + C.\end{aligned}$$

Ex. 6. Find the value of $\int \frac{\sec^2(ax+b)\,dx}{\tan(ax+b)+c}$.

Let $\tan(ax+b)+c=u$. Then $\sec^2(ax+b)\,a\,dx=du$. Hence

$$\begin{aligned}\int \frac{\sec^2(ax+b)\,dx}{\tan(ax+b)+c} &= \int \frac{1}{a}\cdot\frac{du}{u}\\ &= \frac{1}{a}\int\frac{du}{u}\end{aligned}$$

$$= \frac{1}{a}\log u + C$$

$$= \frac{1}{a}\log[\tan(ax+b)+c] + C.$$

Or,

$$\int \frac{\sec^2(ax+b)\,dx}{\tan(ax+b)+c} = \int \frac{1}{a}\cdot\frac{d[\tan(ax+b)+c]}{\tan(ax+b)+c}$$

$$= \frac{1}{a}\int \frac{d[\tan(ax+b)+c]}{\tan(ax+b)+c}$$

$$= \frac{1}{a}\log[\tan(ax+b)+c] + C.$$

The student is advised to use more and more the second method, illustrated in the preceding problems, as he acquires facility in integration.

13. Integrals of trigonometric functions. By rewriting the formulas (§ 7) for the differentiation of the trigonometric functions, we derive the formulas

$$\int \cos u\, du = \sin u, \tag{1}$$

$$\int \sin u\, du = -\cos u, \tag{2}$$

$$\int \sec^2 u\, du = \tan u, \tag{3}$$

$$\int \csc^2 u\, du = -\operatorname{ctn} u, \tag{4}$$

$$\int \sec u \tan u\, du = \sec u, \tag{5}$$

$$\int \csc u \operatorname{ctn} u\, du = -\csc u. \tag{6}$$

In addition to the above are the four following formulas:

$$\int \tan u\, du = \log \sec u, \tag{7}$$

$$\int \operatorname{ctn} u\, du = \log \sin u, \tag{8}$$

$$\int \sec u\, du = \log(\sec u + \tan u) = \log \tan\left(\frac{\pi}{4} + \frac{u}{2}\right), \tag{9}$$

$$\int \csc u\, du = \log(\csc u - \operatorname{ctn} u) = \log \tan \frac{u}{2}. \tag{10}$$

To derive (7) we note that $\tan u = \dfrac{\sin u}{\cos u}$ and that $-\sin u\, du = d(\cos u)$. Then

$$\begin{aligned}\int \tan u\, du &= -\int \frac{d(\cos u)}{\cos u}\\ &= -\log \cos u\\ &= \log \sec u.\end{aligned}$$

In like manner,

$$\int \operatorname{ctn} u\, du = \int \frac{\cos u\, du}{\sin u} = \log \sin u.$$

Direct proofs of (9) and (10) are given in § 68. At present they may be verified by differentiation. For example, (9) is evidently true since

$$d \log(\sec u + \tan u) = \sec u\, du.$$

The second form of the integral may be found by making a trigonometric transformation of $\sec u + \tan u$ to $\tan\left(\dfrac{\pi}{4} + \dfrac{u}{2}\right)$. Formula (10) may be treated in the same manner.

Ex. 1. Find the value of $\int \cos(ax^2 + bx)(2\,ax + b)\, dx$.

Let $ax^2 + bx = u$. Then $(2\,ax + b)\, dx = du$.

Therefore $\int \cos(ax^2 + bx)(2\,ax + b)\, dx = \int \cos(ax^2 + bx)\, d(ax^2 + bx)$

$$= \sin(ax^2 + bx) + C.$$

Ex. 2. Find the value of $\int \sec(e^{ax^2} + b)\tan(e^{ax^2} + b)\, e^{ax^2} x\, dx$.

Let $e^{ax^2} + b = u$. Then $e^{ax^2}\, 2\, ax\, dx = du$.

Therefore $\int \sec(e^{ax^2} + b)\tan(e^{ax^2} + b)\, e^{ax^2} x\, dx$

$$\begin{aligned}&= \frac{1}{2\,a}\int \sec(e^{ax^2} + b)\tan(e^{ax^2} + b)\, d(e^{ax^2} + b)\\ &= \frac{1}{2\,a}\sec(e^{ax^2} + b) + C.\end{aligned}$$

The integral may often be brought under one or more of the fundamental formulas by a trigonometric transformation of the integrand.

Ex. 3. Find the value of $\int \cos^2 x\, dx$.

Since $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$, we have

$$\begin{aligned}\int \cos^2 x\, dx &= \tfrac{1}{2}\int (1 + \cos 2x)\, dx\\ &= \tfrac{1}{2}\int dx + \tfrac{1}{4}\int \cos 2x\, d(2x)\\ &= \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + C.\end{aligned}$$

Ex. 4. Find the value of $\int \sec^6 2x\, dx$.

If we let $\sec^6 2x = \sec^4 2x \cdot \sec^2 2x$, and place $\sec^4 2x = (1 + \tan^2 2x)^2 = 1 + 2\tan^2 2x + \tan^4 2x$, the original integral becomes

$$\int (1 + 2\tan^2 2x + \tan^4 2x)\sec^2 2x\, dx.$$

Place $\tan 2x = u$. Then $\sec^2 2x \cdot 2\, dx = du$. Making this substitution and simplifying, we have

$$\begin{aligned}\int \sec^6 2x\, dx &= \tfrac{1}{2}\int (1 + 2u^2 + u^4)\, du\\ &= \tfrac{1}{2}(u + \tfrac{2}{3}u^3 + \tfrac{1}{5}u^5) + C\\ &= \tfrac{1}{2}\tan 2x + \tfrac{1}{3}\tan^3 2x + \tfrac{1}{10}\tan^5 2x + C.\end{aligned}$$

14. Integrals leading to the inverse trigonometric functions. From the formulas (§ 7) for the differentiation of the inverse trigonometric functions we derive the following corresponding formulas of integration:

$$\int \frac{du}{\sqrt{1 - u^2}} = \sin^{-1} u \quad \text{or} \quad -\cos^{-1} u,$$

$$\int \frac{du}{1 + u^2} = \tan^{-1} u \quad \text{or} \quad -\text{ctn}^{-1} u,$$

$$\int \frac{du}{u\sqrt{u^2 - 1}} = \sec^{-1} u \quad \text{or} \quad -\csc^{-1} u.$$

These formulas are much more serviceable, however, if u is replaced by $\frac{u}{a}$ $(a > 0)$. Making this substitution and evident reductions, we have as our required formulas:

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} \quad \text{or} \quad -\cos^{-1}\frac{u}{a}, \tag{1}$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} \quad \text{or} \quad -\frac{1}{a}\text{ctn}^{-1}\frac{u}{a}, \tag{2}$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} \quad \text{or} \quad -\frac{1}{a}\csc^{-1}\frac{u}{a}. \tag{3}$$

Referring to I, § 153, we see that in (1) $\sin^{-1}\frac{u}{a}$ must be taken in the first or the fourth quadrant and that $\cos^{-1}\frac{u}{a}$ must be taken in the first or the second quadrant. In like manner in (3) $\sec^{-1}\frac{u}{a}$ and $\csc^{-1}\frac{u}{a}$ must be taken in the first or the third quadrant.

It is to be noted that the two results in (1) differ only by a constant. For let $\sin^{-1}\frac{u}{a}=\phi$ and $\cos^{-1}\frac{u}{a}=\psi$, where ϕ is in the first or the fourth quadrant and ψ is in the first or the second quadrant. Then

$$\sin\phi=\frac{u}{a}, \qquad \cos\psi=\frac{u}{a},$$

$$\cos\phi=\sqrt{1-\frac{u^2}{a^2}}, \quad \sin\psi=\sqrt{1-\frac{u^2}{a^2}}.$$

Therefore $\cos(\phi+\psi)=0$, whence $\phi+\psi=(2k+1)\frac{\pi}{2}$, where k is any integer or zero.

Hence $\phi=(2k+1)\frac{\pi}{2}-\psi$, or

$$\sin^{-1}\frac{u}{a}=(2k+1)\frac{\pi}{2}-\cos^{-1}\frac{u}{a}.$$

Similarly, the results in (2) or (3) may be shown to differ by constants.

Ex. 1. Find the value of $\int\frac{dx}{\sqrt{9-4x^2}}$.

Letting $2x=u$, we have $du=2\,dx$, and

$$\int\frac{dx}{\sqrt{9-4x^2}}=\frac{1}{2}\int\frac{d(2x)}{\sqrt{9-(2x)^2}}=\frac{1}{2}\sin^{-1}\frac{2x}{3}+C \text{ or } -\frac{1}{2}\cos^{-1}\frac{2x}{3}+C.$$

Ex. 2. Find the value of $\int\frac{dx}{x\sqrt{3x^2-4}}$.

If we let $\sqrt{3}\,x=u$, then $du=\sqrt{3}\,dx$, and we may write

$$\int\frac{dx}{x\sqrt{3x^2-4}}=\int\frac{d(\sqrt{3}\,x)}{\sqrt{3}\,x\sqrt{(\sqrt{3}\,x)^2-4}}=\frac{1}{2}\sec^{-1}\frac{x\sqrt{3}}{2}+C \text{ or } -\frac{1}{2}\csc^{-1}\frac{x\sqrt{3}}{2}+C.$$

Ex. 3. Find the value of $\int \frac{dx}{\sqrt{4x - x^2}}$.

Since $\sqrt{4x - x^2} = \sqrt{4 - (x-2)^2}$, we have

$$\int \frac{dx}{\sqrt{4x - x^2}} = \int \frac{dx}{\sqrt{4 - (x-2)^2}} = \int \frac{d(x-2)}{\sqrt{4 - (x-2)^2}}$$

$$= \sin^{-1}\frac{x-2}{2} + C \text{ or } -\cos^{-1}\frac{x-2}{2} + C.$$

Ex. 4. Find the value of $\int \frac{dx}{2x^2 + 3x + 5}$.

To avoid fractions and radicals, we place

$$\frac{dx}{2x^2 + 3x + 5} = \frac{8\,dx}{16x^2 + 24x + 40} = 2 \cdot \frac{4\,dx}{(4x+3)^2 + 31}.$$

Therefore

$$\int \frac{dx}{2x^2 + 3x + 5} = 2\int \frac{4\,dx}{(4x+3)^2 + 31} = 2\int \frac{d(4x+3)}{(4x+3)^2 + 31}$$

$$= \frac{2}{\sqrt{31}}\tan^{-1}\frac{4x+3}{\sqrt{31}} + C \text{ or } -\frac{2}{\sqrt{31}}\operatorname{ctn}^{-1}\frac{4x+3}{\sqrt{31}} + C.$$

The methods used in Exs. 3 and 4 are often of value in dealing with functions involving $ax^2 + bx + c$.

Ex. 5. Find the value of $\int \frac{(x^3 + x)\,dx}{5 + 4x^4}$.

Separating the integrand into two fractions, i.e.

$$\frac{x^3}{5 + 4x^4} + \frac{x}{5 + 4x^4},$$

and using § 11 (2) we have

$$\int \frac{(x^3 + x)\,dx}{5 + 4x^4} = \int \frac{x^3\,dx}{5 + 4x^4} + \int \frac{x\,dx}{5 + 4x^4}.$$

But

$$\int \frac{x^3\,dx}{5 + 4x^4} = \frac{1}{16}\int \frac{16x^3\,dx}{5 + 4x^4} = \frac{1}{16}\log(5 + 4x^4);$$

and

$$\int \frac{x\,dx}{5 + 4x^4} = \frac{1}{4}\int \frac{4x\,dx}{5 + (2x^2)^2}$$

$$= \frac{1}{4\sqrt{5}}\tan^{-1}\frac{2x^2}{\sqrt{5}} \text{ or } -\frac{1}{4\sqrt{5}}\operatorname{ctn}^{-1}\frac{2x^2}{\sqrt{5}}.$$

Therefore

$$\int \frac{(x^3 + x)\,dx}{5 + 4x^4} = \frac{1}{16}\log(5 + 4x^4) + \frac{1}{4\sqrt{5}}\tan^{-1}\frac{2x^2}{\sqrt{5}} + C,$$

or

$$\frac{1}{16}\log(5 + 4x^4) - \frac{1}{4\sqrt{5}}\operatorname{ctn}^{-1}\frac{2x^2}{\sqrt{5}} + C.$$

15. Closely resembling formulas (1) and (2) of the last article in the form of the integrand are the following two formulas:

$$\int \frac{du}{\sqrt{u^2+a^2}} = \log\left(u+\sqrt{u^2+a^2}\right) \tag{1}$$

and

$$\int \frac{du}{u^2-a^2} = \frac{1}{2a}\log\frac{u-a}{u+a} \quad \text{or} \quad \frac{1}{2a}\log\frac{a-u}{a+u}. \tag{2}$$

To derive (1) we place $u = a\tan\phi$. Then $du = a\sec^2\phi\, d\phi$, and $\sqrt{u^2+a^2} = a\sec\phi$. Therefore

$$\begin{aligned}\int \frac{du}{\sqrt{u^2+a^2}} &= \int \sec\phi\, d\phi \\ &= \log(\sec\phi+\tan\phi) \qquad \text{(by (9), § 13)} \\ &= \log\left(\frac{u+\sqrt{u^2+a^2}}{a}\right) \\ &= \log\left(u+\sqrt{u^2+a^2}\right)-\log a.\end{aligned}$$

But $\log a$ is a constant, and may accordingly be omitted from the formula of integration. If retained, it would affect the constant of integration only.

Formula (2) is derived by means of the fact that the fraction $\frac{1}{u^2-a^2}$ may be separated into two fractions, the denominators of which are respectively $u-a$ and $u+a$; i.e.

$$\frac{1}{u^2-a^2} = \frac{1}{2a}\left(\frac{1}{u-a}-\frac{1}{u+a}\right). \qquad (\text{§ } 53)$$

Then

$$\begin{aligned}\int \frac{du}{u^2-a^2} &= \frac{1}{2a}\int\left(\frac{1}{u-a}-\frac{1}{u+a}\right)du \\ &= \frac{1}{2a}\left(\int\frac{du}{u-a}-\int\frac{du}{u+a}\right) \\ &= \frac{1}{2a}\left[\log(u-a)-\log(u+a)\right] \\ &= \frac{1}{2a}\log\frac{u-a}{u+a}.\end{aligned}$$

The second form of (2) is derived by noting that

$$\int \frac{du}{u-a} = \int \frac{-\,du}{a-u} = \log(a-u).$$

The two results differ only by a constant, for

$$\frac{a-u}{a+u} = -1 \cdot \frac{u-a}{u+a},$$

and hence $$\log \frac{a-u}{a+u} = \log(-1) + \log \frac{u-a}{u+a};$$

and $\log(-1)$ is a constant complex quantity which can be expressed in terms of i (§ 170).

Ex. 1. Find the value of $\int \frac{dx}{\sqrt{3x^2+4x}}$.

To avoid fractions we multiply both numerator and denominator by $\sqrt{3}$.

Then $$\frac{dx}{\sqrt{3x^2+4x}} = \frac{\sqrt{3}\,dx}{\sqrt{9x^2+12x}} = \frac{\sqrt{3}\,dx}{\sqrt{(3x+2)^2-4}}.$$

Letting $3x+2=u$, we have $du = 3\,dx$, and

$$\begin{aligned}\int \frac{dx}{\sqrt{3x^2+4x}} &= \frac{1}{\sqrt{3}} \int \frac{3\,dx}{\sqrt{(3x+2)^2-4}} \\ &= \frac{1}{\sqrt{3}} \log\left(3x+2+\sqrt{(3x+2)^2-4}\right) + C \\ &= \frac{1}{\sqrt{3}} \log\left(3x+2+\sqrt{9x^2+12x}\right) + C.\end{aligned}$$

Ex. 2. Find the value of $\int \frac{dx}{2x^2+x-15}$.

Multiplying the numerator and the denominator by 8, we have

$$\begin{aligned}\int \frac{dx}{2x^2+x-15} &= 2\int \frac{4\,dx}{(4x+1)^2-(11)^2} \\ &= \frac{1}{11} \log \frac{(4x+1)-11}{(4x+1)+11} + C.\end{aligned}$$

This may be reduced to $\frac{1}{11}\log\frac{2x-5}{2x+6}+C$, or $\frac{1}{11}\log\frac{2x-5}{x+3}-\frac{1}{11}\log 2+C$, and the term $-\frac{1}{11}\log 2$, being independent of x, may be omitted, as it will only affect the value of the constant of integration.

If in the formulas for the differentiation of

$$\sinh^{-1} u, \ \cosh^{-1} u, \ \text{and} \ \tanh^{-1} u \qquad \text{(I, § 161)}$$

we replace u by $\frac{u}{a}$, they become

$$\frac{d}{dx}\sinh^{-1}\frac{u}{a} = \frac{\frac{du}{dx}}{\sqrt{u^2+a^2}},$$

$$\frac{d}{dx}\cosh^{-1}\frac{u}{a} = \frac{\frac{du}{dx}}{\sqrt{u^2-a^2}},$$

and

$$\frac{d}{dx}\tanh^{-1}\frac{u}{a} = \frac{a\frac{du}{dx}}{a^2-u^2}.$$

The corresponding formulas of integration are evidently

$$\int\frac{du}{\sqrt{u^2+a^2}} = \sinh^{-1}\frac{u}{a},$$

$$\int\frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\frac{u}{a},$$

and

$$\int\frac{du}{a^2-u^2} = \frac{1}{a}\tanh^{-1}\frac{u}{a}.$$

These forms of the integrals are often of advantage in problems where the resulting equation has to be solved for the value of u in terms of the other quantities in the equation.

By formulas (2), § 14, and (2) of this article we can find the value of any integral of the form $\int\frac{dx}{ax^2+bx+c}$. We can also find the value of any integral of the form $\int\frac{(Ax+B)\,dx}{ax^2+bx+c}$, as shown in Ex. 3. Finally, we note that the value of $\int\frac{P(x)\,dx}{ax^2+bx+c}$, where $P(x)$ is a polynomial, can always be found, since by division the integrand is equal to a polynomial plus a fraction of the form just mentioned.

Ex. 3. Find the value of $\int \frac{(3x+4)\,dx}{2x^2+x-15}$.

If $2x^2+x-15=u$, $du=(4x+1)\,dx$.
Now $3x+4$ may be written as $\frac{3}{4}(4x+1)+\frac{13}{4}$.

$$\text{Therefore}\quad \int \frac{(3x+4)\,dx}{2x^2+x-15} = \int \frac{[\frac{3}{4}(4x+1)+\frac{13}{4}]\,dx}{2x^2+x-15}$$
$$= \frac{3}{4}\int \frac{(4x+1)\,dx}{2x^2+x-15} + \frac{13}{4}\int \frac{dx}{2x^2+x-15}.$$

The first integral is $\frac{3}{4}\log(2x^2+x-15)$, by § 12 (2), and the last integral is of the form solved in Ex. 2, and is $\frac{13}{44}\log\frac{2x-5}{x+3}$.

Hence the complete integral is

$$\frac{3}{4}\log(2x^2+x-15)+\frac{13}{44}\log\frac{2x-5}{x+3}+C.$$

Ex. 4. Find the value of $\int \frac{(2x+5)\,dx}{\sqrt{3x^2+4x}}$.

The value of this integral may be made to depend upon that of Ex. 1 in the same way that the solution of Ex. 3 was made to depend upon the solution of Ex. 2. For let $3x^2+4x=u$; then $du=(6x+4)\,dx$.

Now $2x+5=\frac{1}{3}(6x+4)+\frac{11}{3}$.

$$\text{Therefore}\quad \int \frac{(2x+5)\,dx}{\sqrt{3x^2+4x}} = \int \frac{[\frac{1}{3}(6x+4)+\frac{11}{3}]\,dx}{\sqrt{3x^2+4x}}$$
$$= \frac{1}{3}\int (3x^2+4x)^{-\frac{1}{2}}[(6x+4)\,dx] + \frac{11}{3}\int \frac{dx}{\sqrt{3x^2+4x}}.$$

The first integral is $\frac{2}{3}\sqrt{3x^2+4x}$, by § 12 (1), and the second integral is $\frac{11}{3\sqrt{3}}\log(3x+2+\sqrt{9x^2+12x})$, by Ex. 1. Hence the complete integral is

$$\frac{2}{3}\sqrt{3x^2+4x}+\frac{11}{3\sqrt{3}}\log(3x+2+\sqrt{9x^2+12x})+C.$$

16. Integrals of exponential functions. The formulas

$$\int e^u\,du = e^u \qquad (1)$$

and

$$\int a^u\,du = \frac{1}{\log a}\,a^u \qquad (2)$$

are derived immediately from the corresponding formulas of differentiation. The proof is left to the student.

17. Collected formulas.

$$\int u^n du = \frac{u^{n+1}}{n+1}, \tag{1}$$

$$\int \frac{du}{u} = \log u, \tag{2}$$

$$\int \cos u\, du = \sin u, \tag{3}$$

$$\int \sin u\, du = -\cos u, \tag{4}$$

$$\int \sec^2 u\, du = \tan u, \tag{5}$$

$$\int \csc^2 u\, du = -\operatorname{ctn} u, \tag{6}$$

$$\int \sec u \tan u\, du = \sec u, \tag{7}$$

$$\int \csc u \operatorname{ctn} u\, du = -\csc u, \tag{8}$$

$$\int \tan u\, du = \log \sec u, \tag{9}$$

$$\int \operatorname{ctn} u\, du = \log \sin u, \tag{10}$$

$$\int \sec u\, du = \log(\sec u + \tan u) = \log \tan\left(\frac{\pi}{4} + \frac{u}{2}\right), \tag{11}$$

$$\int \csc u\, du = \log(\csc u - \operatorname{ctn} u) = \log \tan \frac{u}{2}, \tag{12}$$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} \text{ or } -\cos^{-1}\frac{u}{a}, \tag{13}$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} \text{ or } -\frac{1}{a}\operatorname{ctn}^{-1}\frac{u}{a}, \tag{14}$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} \text{ or } -\frac{1}{a}\csc^{-1}\frac{u}{a}, \tag{15}$$

$$\int \frac{du}{\sqrt{u^2 + a^2}} = \log(u + \sqrt{u^2 + a^2}) \text{ or } \sinh^{-1}\frac{u}{a}, \tag{16}$$

$$\int \frac{du}{\sqrt{u^2 - a^2}} = \log(u + \sqrt{u^2 - a^2}) \text{ or } \cosh^{-1}\frac{u}{a}, \tag{17}$$

$$\int\frac{du}{u^2-a^2}=\frac{1}{2a}\log\frac{u-a}{u+a}\quad\text{or}\quad\frac{1}{2a}\log\frac{a-u}{a+u}\quad\text{or}\quad-\frac{1}{a}\tanh^{-1}\frac{u}{a},\tag{18}$$

$$\int e^u\,du=e^u,\tag{19}$$

$$\int a^u\,du=\frac{1}{\log a}\,a^u.\tag{20}$$

18. Integration by substitution. In order to evaluate a given integral it is necessary to reduce it to one of the foregoing standard forms. A very important method by which this may be done is that of the *substitution* of a new variable. In fact, the work thus far has been of this nature, in that by inspection we have taken some function of x as u.

In many cases where the substitution is not so obvious as in the previous examples, it is still possible by the proper choice of a new variable to reduce the integral to a known form. The choice of the new variable depends largely upon the skill and the experience of the worker, and no rules can be given to cover all cases. A systematic discussion of some types of desirable substitution will be taken up in later chapters, but we shall in this chapter work a few illustrative examples.

Ex. 1. Find the value of $\int\frac{x^2\,dx}{\sqrt{2x+3}}$.

Let $2x+3=z^2$. Then $x=\frac{1}{2}(z^2-3)$ and $dx=z\,dz$. Substituting these values in the original integral, we have

$$\tfrac{1}{4}\int(z^4-6z^2+9)\,dz=\tfrac{1}{4}\left(\tfrac{1}{5}z^5-2z^3+9z\right)+C.$$

Replacing z by its value in terms of x, we have

$$\int\frac{x^2\,dx}{\sqrt{2x+3}}=\frac{1}{5}\sqrt{2x+3}\,(x^2-2x+6)+C.$$

Ex. 2. Find the value of $\int\frac{\sqrt{x^2+a^2}}{x}\,dx$.

Let $x^2+a^2=z^2$. Then $x\,dx=z\,dz$, and $\frac{dx}{x}=\frac{1}{x^2}\cdot x\,dx=\frac{z\,dz}{z^2-a^2}$. Then, after substitution, we have

$$\int\frac{z^2\,dz}{z^2-a^2}=\int\left(1+\frac{a^2}{z^2-a^2}\right)dz=z+\frac{a}{2}\log\frac{z-a}{z+a}+C.$$

Replacing z by its value in terms of x, we have

$$\int \frac{\sqrt{x^2+a^2}}{x}\,dx = \sqrt{x^2+a^2} + \frac{a}{2}\log\frac{\sqrt{x^2+a^2}-a}{\sqrt{x^2+a^2}+a} + C.$$

Ex. 3. Find the value of $\int \sqrt{a^2-x^2}\,dx$.

Let $x = a\sin z$. Then $dx = a\cos z\,dz$ and $\sqrt{a^2-x^2} = a\cos z$.

Therefore $\int \sqrt{a^2-x^2}\,dx = a^2\int \cos^2 z\,dz = \frac{1}{2}a^2\int (1+\cos 2z)\,dz$

$$= \tfrac{1}{2}a^2\left(z + \tfrac{1}{2}\sin 2z\right) + C.$$

But $z = \sin^{-1}\dfrac{x}{a}$, and $\sin 2z = 2\sin z\cos z = 2\dfrac{x}{a^2}\sqrt{a^2-x^2}$.

Finally, by substitution, we have

$$\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left(x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}\right) + C.$$

Ex. 4. Find the value of $\int x^3\sqrt{x^2-a^2}\,dx$.

Let $x = a\sec z$. Then $dx = a\sec z\tan z\,dz$, and $\sqrt{x^2-a^2} = a\tan z$.

Therefore $\int x^3\sqrt{x^2-a^2}\,dx = a^5\int \tan^2 z\sec^4 z\,dz$

$$= a^5\int (\tan^2 z + \tan^4 z)\sec^2 z\,dz$$

$$= a^5\left(\tfrac{1}{3}\tan^3 z + \tfrac{1}{5}\tan^5 z\right) + C.$$

But $\sec z = \dfrac{x}{a}$, whence $\tan z = \dfrac{\sqrt{x^2-a^2}}{a}$, so that, by substitution, we have

$$\int x^3\sqrt{x^2-a^2}\,dx = \tfrac{1}{15}\sqrt{(x^2-a^2)^3}\,(2a^2+3x^2) + C.$$

Ex. 5. Find the value of $\int \dfrac{dx}{(x^2+a^2)^{\frac{3}{2}}}$.

Let $x = a\tan z$. Then $dx = a\sec^2 z\,dz$ and $\sqrt{x^2+a^2} = a\sec z$.

Therefore $\int \dfrac{dx}{(x^2+a^2)^{\frac{3}{2}}} = \dfrac{1}{a^2}\int \dfrac{dz}{\sec z} = \dfrac{1}{a^2}\int \cos z\,dz = \dfrac{1}{a^2}\sin z + C.$

But $\tan z = \dfrac{x}{a}$, whence $\sin z = \dfrac{x}{\sqrt{x^2+a^2}}$, so that, by substitution,

$$\int \frac{dx}{(x^2+a^2)^{\frac{3}{2}}} = \frac{x}{a^2\sqrt{x^2+a^2}} + C.$$

Ex. 6. Find the value of $\int \frac{dx}{(2x+1)\sqrt{5x^2+8x+3}}$.

Let $2x+1=\frac{1}{z}$. Then $x=\frac{1}{2}\left(\frac{1}{z}-1\right)$, $dx=-\frac{1}{2z^2}dz$, and $\sqrt{5x^2+8x+3}$ $=\frac{1}{2z}\sqrt{z^2+6z+5}$.

Therefore

$$\int \frac{dx}{(2x+1)\sqrt{5x^2+8x+3}}=-\int \frac{dz}{\sqrt{z^2+6z+5}}=-\int \frac{dz}{\sqrt{(z+3)^2-4}}$$
$$=-\log\left(z+3+\sqrt{z^2+6z+5}\right)+C.$$

But $z=\frac{1}{2x+1}$, and hence

$$-\log\left(z+3+\sqrt{z^2+6z+5}\right)=-\log\frac{6x+4+2\sqrt{5x^2+8x+3}}{2x+1}$$
$$=\log\frac{2x+1}{3x+2+\sqrt{5x^2+8x+3}}-\log 2.$$

$$\therefore \int \frac{dx}{(2x+1)\sqrt{5x^2+8x+3}}=\log\frac{2x+1}{3x+2+\sqrt{5x^2+8x+3}}+C,$$

$-\log 2$ having been made a part of the constant of integration.

The student should refer freely to these examples as possibly suggesting a type of substitution desirable in the solution of a new problem. From them the following hints for substitution in similar cases may be deduced:

In integrals involving $\sqrt[n]{a+bx}$ try $a+bx=z^n$, as in Ex. 1.

In integrals involving $\sqrt{x^2+a^2}$ try either $x^2+a^2=z^2$, as in Ex. 2, or $x=a\tan z$, as in Ex. 5.

In integrals involving $\sqrt{a^2-x^2}$ try $x=a\sin z$, as in Ex. 3.

In integrals involving $\sqrt{x^2-a^2}$ try $x=a\sec z$, as in Ex. 4.

In integrals of the form $\int \frac{dx}{(Ax+B)\sqrt{ax^2+bx+c}}$ try $Ax+B=\frac{1}{z}$, as in Ex. 6.

It is not to be supposed that the above substitutions are desirable in all cases. For instance, in Ex. 2 the substitution $x=a\tan z$ does not simplify the integral; but the substitution $x^2+a^2=z^2$ is of advantage, though it is rare that the substitution of a single letter for the square root of a quadratic polynomial leads to any simplification.

19. Integration by parts. Another method of importance in the reduction of a given integral to a known type is that of *integration by parts,* the formula for which is derived from the formula for the differential of a product,

$$d(uv) = u\,dv + v\,du.$$

From this formula we derive directly that

$$uv = \int u\,dv + \int v\,du,$$

which is usually written in the form

$$\int u\,dv = uv - \int v\,du.$$

In the use of this formula the aim is evidently to make the original integration depend upon the evaluation of a simpler integral.

Ex. 1. Find the value of $\int xe^x dx$.

If we let $x = u$ and $e^x dx = dv$, we have $du = dx$ and $v = e^x$. Substituting in our formula, we have

$$\begin{aligned}\int xe^x dx &= xe^x - \int e^x dx \\ &= xe^x - e^x + C \\ &= (x-1)\,e^x + C.\end{aligned}$$

It is evident that in selecting the expression for dv it is desirable, if possible, to choose an expression that is easily integrated.

Ex. 2. Find the value of $\int \sin^{-1}x\,dx$.

Here we may let $\sin^{-1}x = u$ and $dx = dv$, whence $du = \dfrac{dx}{\sqrt{1-x^2}}$ and $v = x$. Substituting in our formula, we have

$$\begin{aligned}\int \sin^{-1}x\,dx &= x\sin^{-1}x - \int \frac{x\,dx}{\sqrt{1-x^2}} \\ &= x\sin^{-1}x + \sqrt{1-x^2} + C,\end{aligned}$$

the last integral being evaluated by § 12 (1).

Ex. 3. Find the value of $\int x\cos^2 x\,dx$.

Since $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$, we have

$$\int x\cos^2 x\,dx = \frac{1}{2}\int (x + x\cos 2x)\,dx = \frac{x^2}{4} + \frac{1}{2}\int x\cos 2x\,dx.$$

Letting $x = u$ and $\cos 2x\,dx = dv$, we have $du = dx$ and $v = \frac{1}{2}\sin 2x$.

Therefore
$$\int x\cos 2x\,dx = \frac{x}{2}\sin 2x - \frac{1}{2}\int \sin 2x\,dx$$
$$= \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + C.$$

$$\therefore \int x\cos^2 x\,dx = \frac{x^2}{4} + \frac{1}{2}\left(\frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x\right) + C$$
$$= \tfrac{1}{8}(2x^2 + 2x\sin 2x + \cos 2x) + C.$$

Sometimes an integral may be evaluated by successive integration by parts.

Ex. 4. Find the value of $\int x^2 e^x\,dx$.

Here we will let $x^2 = u$ and $e^x dx = dv$. Then $du = 2x\,dx$ and $v = e^x$.

Therefore
$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx.$$

The integral $\int x e^x\,dx$ may be evaluated by integration by parts (see Ex. 1), so that finally
$$\int x^2 e^x\,dx = x^2 e^x - 2(x-1)e^x + C = e^x(x^2 - 2x + 2) + C.$$

Ex. 5. Find the value of $\int e^{ax}\sin bx\,dx$.

Letting $\sin bx = u$ and $e^{ax}dx = dv$, we have
$$\int e^{ax}\sin bx\,dx = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\int e^{ax}\cos bx\,dx.$$

In the integral $\int e^{ax}\cos bx\,dx$ we let $\cos bx = u$ and $e^{ax}dx = dv$, and have
$$\int e^{ax}\cos bx\,dx = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\,dx.$$

Substituting this value above, we have
$$\int e^{ax}\sin bx\,dx = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\left(\frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\,dx\right).$$

Now bringing to the left-hand member of the equation all the terms containing the integral, we have
$$\left(1 + \frac{b^2}{a^2}\right)\int e^{ax}\sin bx\,dx = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}e^{ax}\cos bx,$$

whence
$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2 + b^2}.$$

Ex. 6. Find the value of $\int e^{ax} \cos bx \, dx$.

The result is $\dfrac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}$, the work being left to the student, since it is exactly like that of Ex. 5.

Ex. 7. Find the value of $\int \sqrt{x^2 + a^2}\, dx$.

Placing $\sqrt{x^2 + a^2} = u$ and $dx = dv$, whence $du = \dfrac{x\, dx}{\sqrt{x^2 + a^2}}$ and $v = x$, we have

$$\int \sqrt{x^2 + a^2}\, dx = x\sqrt{x^2 + a^2} - \int \frac{x^2\, dx}{\sqrt{x^2 + a^2}}. \tag{1}$$

Since $x^2 = (x^2 + a^2) - a^2$, the second integral of (1) may be written as

$$\int \frac{(x^2 + a^2)\, dx}{\sqrt{x^2 + a^2}} - a^2 \int \frac{dx}{\sqrt{x^2 + a^2}},$$

which equals

$$\int \sqrt{x^2 + a^2}\, dx - a^2 \int \frac{dx}{\sqrt{x^2 + a^2}}.$$

Evaluating this last integral and substituting in (1), we have

$$\int \sqrt{x^2 + a^2}\, dx = x\sqrt{x^2 + a^2} - \int \sqrt{x^2 + a^2}\, dx + a^2 \log\left(x + \sqrt{x^2 + a^2}\right),$$

whence

$$\int \sqrt{x^2 + a^2}\, dx = \tfrac{1}{2}\left[x\sqrt{x^2 + a^2} + a^2 \log\left(x + \sqrt{x^2 + a^2}\right)\right].$$

20. Possibility of integration. In this chapter we have learned how to express the integrals of certain types of functions in terms of the elementary functions, and the discussion of methods of integration will be continued in Chaps. VI and VII. But it should be noted now that it is not always possible to express the integral of elementary functions in terms of elementary functions. For example, $\displaystyle\int \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}}$ cannot be so expressed; in fact, this integral defines a function of x of an entirely new kind.

Accordingly when it is said that the integration of certain functions is not possible, it is meant that the integration is not possible for one who knows only the elementary functions, which are in fact the functions generally used in applied mathematics. In this respect integration differs radically from differentiation, which can always be performed upon elementary functions. This fact is not surprising, since it is closely analogous to what takes place in connection with other operations which are the inverse of each other. For example, the addition of positive numbers can always be expressed in terms of positive numbers, but the inverse operation of subtraction is made always possible only by the introduction of negative numbers; also the involution of rational numbers is always possible, but evolution is always possible only after irrational numbers are introduced.

PROBLEMS

Find the values of the following integrals:

1. $\int (3x^2 + 6x + 1)\,dx.$
2. $\int \left(x^2 + 2 + \frac{1}{x^2}\right) dx.$
3. $\int \left(\sqrt{x^3} + \frac{1}{\sqrt{x^3}}\right) dx.$
4. $\int \frac{x + \sqrt[3]{x^2} + 5}{\sqrt{x}}\,dx.$
5. $\int \frac{3 + x^2}{\sqrt[3]{x}}\,dx.$
6. $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^3 dx.$
7. $\int (x - 1)^2 x^2\,dx.$
8. $\int \frac{(x^2 - 1)^3}{x}\,dx.$
9. $\int \frac{x^3\,dx}{x + 2}.$
10. $\int \sqrt{1 + e^x}\, e^x\,dx.$
11. $\int \frac{e^x\,dx}{e^x + a}.$
12. $\int \frac{dx}{x \log x}.$
13. $\int \frac{dx}{(1 + x^2)\tan^{-1}x}.$
14. $\int \frac{1 + \cos x}{(x + \sin x)^3}\,dx.$
15. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx.$
16. $\int \sqrt{a + bx}\,dx.$
17. $\int \frac{(2 - 4x)\,dx}{3 - 4x + 4x^2}.$
18. $\int (2 - 3x)^2 dx.$
19. $\int \frac{\sin x\,dx}{a + b\cos x}.$
20. $\int \frac{x\,dx}{(1 + x^2)^3}.$
21. $\int \frac{3x\,dx}{(2 + 3x^2)^2}.$
22. $\int \frac{(x + 2)\,dx}{\sqrt{5 + 4x + x^2}}.$
23. $\int \frac{a + bx}{a' + b'x}\,dx.$
24. $\int \frac{e^{2x} + \sec^2 2x}{\epsilon^{2x} + \tan 2x}\,dx$
25. $\int \frac{dx}{(x + a)[\log(x + a)]^n}.$
26. $\int \frac{(\tan^{-1}x - 1)^{\frac{1}{2}}}{1 + x^2}\,dx.$
27. $\int \frac{\log x}{x}\,dx.$

28. $\int (e^{ax^2}+b)^2 e^{ax^2} x\,dx.$

29. $\int \left(\sqrt{\frac{a+x}{a-x}}-\sqrt{\frac{a-x}{a+x}}\right)dx.$

30. $\int \frac{dx}{1+e^x}.$

31. $\int \frac{dx}{\sqrt{x^2+a^2}\log\left(x+\sqrt{x^2+a^2}\right)}.$

32. $\int \frac{x\sqrt{\log(x^2+a^2)}}{x^2+a^2}dx.$

33. $\int \frac{5\sqrt[3]{x^2}-4\sqrt[3]{x}}{\sqrt[3]{x^5}-\sqrt[3]{x^4}}dx.$

34. $\int \frac{a\sqrt{bx}-b\sqrt{ax}}{x[\sqrt{ax}-\sqrt{bx}]^3}dx.$

35. $\int \sin^3 x \cos x\,dx.$

36. $\int [\sin^4(ax+b)+\cos^4(ax+b)]\sin(ax+b)\cos(ax+b)\,dx.$

37. $\int (\csc bx - \operatorname{ctn} bx)\csc bx\,dx.$

38. $\int \frac{dx}{\sin^2(ax+b)}.$

39. $\int \frac{\sec^2(x^2+a^2)\cdot x\,dx}{\tan^3(x^2+a^2)}.$

40. $\int \frac{\sec x^2 \tan x^2 \cdot x\,dx}{a^2+\sec x^2}.$

41. $\int (\sec x\tan x+\sec x)^2 dx.$

42. $\int \sin^2 2x\,dx.$

43. $\int \frac{\cos 2x\,dx}{\sin x}.$

44. $\int \cos x\sin 2x\,dx.$

45. $\int (\tan^2 x-\operatorname{ctn}^2 x)\,dx.$

46. $\int (\sec 3x-\tan 3x)^2 dx.$

47. $\int \frac{\sec^3 x+\tan^3 x}{\sec x+\tan x}dx.$

48. $\int \sin mx\sin nx\,dx,\ (m\neq n).$

49. $\int \cos mx\cos nx\,dx,\ (m\neq n).$

50. $\int \sin(ax+b)\cos(a'x+b')\,dx.$

51. $\int (\tan 2x+\operatorname{ctn} 2x)^2 dx.$

52. $\int (\sec x^2+\tan x^2)^2\sec x^2\cdot x\,dx.$

53. $\int \cos^2(1-2x)\,dx.$

54. $\int \left(\frac{\sin 2x}{\cos x}+\frac{\cos x}{\sin 2x}\right)dx.$

55. $\int (\sec 2x+\tan 2x-1)(\sec 2x-\tan 2x-1)\,dx.$

56. $\int \sqrt{\frac{\csc x+\operatorname{ctn} x}{\csc x-\operatorname{ctn} x}}dx.$

57. $\int \frac{\cos 2\theta\,d\theta}{\cos\theta-\sin\theta}.$

58. $\int \frac{\sec\theta\,d\theta}{\sec\theta+\tan\theta}.$

59. $\int \frac{\sin\theta\,d\theta}{1-\sin\theta}.$

60. $\int \frac{1-\cos x}{1+\cos x}dx.$

61. $\int \frac{dx}{1+\cos x}.$

62. $\int \frac{\operatorname{ctn} x\,dx}{\sin x-1}.$

63. $\int \frac{dx}{9x^2+25}.$

64. $\int \frac{dx}{\sqrt{3-4x^2}}.$

65. $\int \frac{dx}{x\sqrt{25x^2-1}}.$

66. $\int \frac{dx}{\sqrt{1-3x^2}}.$

67. $\int \frac{dx}{2x^2+1}.$

68. $\int \frac{dx}{x\sqrt{5x^2-3}}.$

69. $\int \frac{dx}{x^2-2x+10}.$

70. $\int \frac{x^2\,dx}{\sqrt{a^6-x^6}}.$

71. $\int \frac{dx}{x\sqrt{x^4-a^4}}.$

72. $\int \frac{dx}{13-6x+x^2}.$

73. $\int \frac{dx}{\sqrt{1+2x\tan\alpha-x^2}}.$

74. $\int \frac{dx}{x\sqrt{a^2x^2-b^2}}.$

75. $\int \frac{dx}{x^2+2x\sin\alpha+1}.$

76. $\int \frac{dx}{\sqrt{4x-x^2}}.$

77. $\int \frac{e^x\,dx}{e^{2x}+2e^x\tan\alpha+\sec^2\alpha}.$

78. $\int \frac{dx}{(x-2)\sqrt{x^2-4x+2}}.$

79. $\int \frac{dx}{\sqrt{2x-3x^2}}.$

80. $\int \frac{e^x\,dx}{\sqrt{1-2e^x\operatorname{ctn}\alpha-e^{2x}}}.$

81. $\int \frac{dx}{2x^2-2x+1}.$

82. $\int \frac{dx}{(2x+5)\sqrt{x^2+5x+4}}.$

83. $\int \frac{dx}{x^2\sin^2\alpha+x\sin 2\alpha+1}.$

84. $\int \frac{x\,dx}{\sqrt{3-6x^2-x^4}}.$

85. $\int \frac{dx}{\sqrt{-x^2-3x+4}}.$

86. $\int \frac{dx}{(3x-1)\sqrt{3x^2-2x-8}}.$

87. $\int \frac{dx}{\sqrt{-2x^2-7x+4}}.$

88. $\int \frac{2+(1+x^2)x}{1+x^2}\,dx.$

89. $\int \frac{3x+2}{2x^2+1}\,dx.$

90. $\int \frac{x^3-7x}{3x^4+7}\,dx.$

91. $\int \frac{\sqrt{a+x}}{\sqrt{a-x}}\,dx.$

92. $\int \left(\sqrt{\frac{1+x}{1-x}}+\sqrt{\frac{1-x}{1+x}}\right)dx.$

93. $\int \frac{\sin x\,dx}{1+\cos^2 x}.$

94. $\int \frac{dx}{a^2+b^2\cos^2 x}.$

95. $\int \frac{dx}{4x^2-9}.$

96. $\int \frac{dx}{\sqrt{9x^2+2}}.$

97. $\int \frac{dx}{3x^2-1}.$

98. $\int \frac{dx}{\sqrt{4x^2-3}}.$

99. $\int \frac{dx}{\sqrt{2x^2-1}}.$

100. $\int \frac{dx}{4-18x^2}.$

101. $\int \frac{x^2\,dx}{\sqrt{x^6-a^6}}.$

102. $\int \frac{x\,dx}{x^4+4x^2+2}.$

103. $\int \frac{dx}{\sqrt{x^2-2x\tan\alpha-1}}.$

104. $\int \frac{dx}{2x^2+3x+1}.$

105. $\int \frac{dx}{\sqrt{x^2+2x\sin\alpha+1}}.$

106. $\int \frac{dx}{3x^2-6x+2}.$

107. $\int \frac{dx}{\sqrt{5x^2-3x-3}}.$

108. $\int \frac{dx}{15+2x-x^2}.$

109. $\int \frac{dx}{\sqrt{3x^2-2x+1}}.$

110. $\int \frac{dx}{3x-1-2x^2}.$

111. $\int \frac{dx}{\sqrt{2x^2+4x-5}}.$

112. $\int \frac{3x^3+2x}{\sqrt{x^4-a^4}}dx.$

113. $\int \frac{(2x+6)\,dx}{2x^2+3x+1}.$

114. $\int \frac{(12x+2)\,dx}{9x^2-6x-3}.$

115. $\int \frac{(2x-3)\,dx}{\sqrt{2+x+x^2}}.$

116. $\int \frac{(3x+7)\,dx}{\sqrt{x^2+4x-1}}.$

117. $\int \frac{(4x-2)\,dx}{3x^2-4x+4}.$

118. $\int \frac{(5x+4)\,dx}{x^2+4x+1}.$

119. $\int \frac{(1-2x)\,dx}{\sqrt{1-4x-x^2}}.$

120. $\int \frac{(3x+5)\,dx}{\sqrt{1+2x-2x^2}}.$

121. $\int \frac{x\,dx}{4x^2+4x\sec\alpha+\tan^2\alpha}.$

122. $\int \frac{(3x+8)\,dx}{\sqrt{1-2x-x^2}}.$

123. $\int \frac{(3x+5)\,dx}{2x^2+6x+7}.$

124. $\int \frac{\sqrt{x^2-a^2}\,dx}{x\sqrt{x^2+a^2}}.$

125. $\int \frac{\sqrt{x+1}\,dx}{x\sqrt{x-1}}.$

126. $\int (a^x+x^a)\,dx.$

127. $\int (e^x+e^{-x})^3\,dx.$

128. $\int (a^{x^2}+a^{-x^2})^2 x\,dx.$

129. $\int e^{x^2+1}x\,dx.$

130. $\int \frac{dx}{x^2\sqrt[x]{e}}.$

131. $\int a^{\cos 2x}\sin x\cos x\,dx.$

132. $\int \frac{e^{\tan^{-1}x}\,dx}{1+x^2}.$

133. $\int e^{b+cx}a^{b+cx}\,dx.$

134. $\int (a^{nx}+b^{mx})^2\,dx.$

135. $\int \frac{xe^{\sin^{-1}x^2}}{\sqrt{1-x^4}}dx.$

136. $\int \frac{\csc x(\csc x\log a+\operatorname{ctn} x)}{a^{\operatorname{ctn} x}e^{\csc x}}dx.$

137. $\int \frac{a^x\,dx}{\sqrt{a^{2x}+2a^x+\sec^2\alpha}}.$

138. $\int \frac{a^x\,dx}{\sqrt{4a^x-a^{2x}}}.$

139. $\int \frac{dx}{a^x+a^{-x}}.$

140. $\int \frac{e^{\frac{x}{a}}+2}{e^{\frac{x}{a}}+1}dx.$

141. $\int \frac{e^x-1}{e^x+1}dx.$

142. $\int \frac{dx}{x^2 \sqrt{a^2 - x^2}}$.

143. $\int \frac{dx}{\sqrt{(a^2 - x^2)^3}}$.

144. $\int \frac{\sqrt{x^2 - a^2}}{x} dx$.

145. $\int \frac{x^2 \, dx}{\sqrt{(a^2 - x^2)^3}}$.

146. $\int \frac{dx}{x^3 \sqrt{x^2 - a^2}}$.

147. $\int \frac{x^3 \, dx}{(a^2 - x^2)^{\frac{5}{2}}}$.

148. $\int x^3 \sqrt{a^2 + x^2} \, dx$.

149. $\int \frac{dx}{(x^2 - a^2)^{\frac{3}{2}}}$.

150. $\int \frac{dx}{x \sqrt{a^2 + x^2}}$.

151. $\int \frac{x^3 \, dx}{\sqrt{a^2 + x^2}}$.

152. $\int \frac{dx}{\sqrt{e^{2ax} - b^2}}$.

153. $\int \frac{\sqrt{x} + 1}{\sqrt{x} - 1} dx$.

154. $\int x^2 \sqrt{2x + 3} \, dx$.

155. $\int \frac{x^2 \, dx}{(a + bx)^3}$.

156. $\int \frac{x^2 \, dx}{\sqrt{x - 1}}$.

157. $\int x (x + a)^{\frac{1}{3}} \, dx$.

158. $\int \frac{dx}{x \sqrt{x^2 - a^2}}$.

159. $\int \frac{x^5 \, dx}{\sqrt{x^2 + a^2}}$.

160. $\int \frac{x^7 \, dx}{2 + 5x^4}$.

161. $\int \frac{6x^5 \, dx}{(5 - 7x^3)^3}$.

162. $\int \frac{x^3 \, dx}{(1 + x^2)^3}$.

163. $\int \frac{x^3 \, dx}{(x^2 + 4)^2}$.

164. $\int \frac{x^5 \, dx}{\sqrt[3]{3 + 2x^3}}$.

165. $\int \frac{dx}{x \sqrt{4x^2 - 4x - 1}}$.

166. $\int \frac{dx}{x \sqrt{x^2 + 2ax - a^2}}$.

167. $\int \frac{dx}{(x + 2) \sqrt{x^2 + 8x + 13}}$.

168. $\int \frac{dx}{(x - 1) \sqrt{2x^2 - 2x - 1}}$.

169. $\int \frac{dx}{(x + 3) \sqrt{14 + 2x - x^2}}$.

170. $\int \frac{dx}{x + 4x^6}$.

171. $\int \frac{x^2 \, dx}{(1 + 3x^3)^2}$.

172. $\int \frac{dx}{x^2 (x + 4x^6)^{\frac{2}{3}}}$.

173. $\int \frac{x \, dx}{(2x - x^3)^{\frac{4}{3}}}$.

174. $\int \log ax \, dx$.

175. $\int x^m \log ax \, dx$.

176. $\int \tan^{-1} ax \, dx$.

177. $\int \frac{\log (\log x)}{x} dx$.

178. $\int \log (x + \sqrt{x^2 + a^2})\, dx.$

179. $\int x \sin ax\, dx.$

180. $\int \sec^{-1} ax\, dx.$

181. $\int x \sec^{-1} ax\, dx.$

182. $\int x \tan^{-1} ax\, dx.$

183. $\int x^2 e^{ax}\, dx.$

184. $\int (x + 1)^2 e^x\, dx.$

185. $\int x^2 \cos ax\, dx.$

186. $\int x^3 \sin ax\, dx.$

187. $\int (\log ax)^2\, dx.$

188. $\int (x \log ax)^2\, dx.$

189. $\int x \sin^2 ax\, dx.$

190. $\int e^{2x} \cos^2 x\, dx.$

191. $\int e^x \sin 2x \sin x\, dx.$

192. $\int e^{3x} \sin 3x \cos x\, dx.$

193. $\int x \sin^{-1} ax\, dx.$

CHAPTER III

DEFINITE INTEGRALS

21. Definition. We have said in § 3 that there are two important problems in the use of infinitesimals, the one involving the limit of a quotient, the other the limit of a sum. We have also noted in § 4 that the derivative of a function is the limit of the quotient of two infinitesimal increments, so that the limit of a quotient is fundamental in the differential calculus and its applications.

Similarly, there exists a limit of a sum which has fundamental connection with the subject of integration. This limit is called a *definite integral* and is defined as follows:

If $f(x)$ is a function of x which is continuous and one-valued for all values of x between $x = a$ and $x = b$ inclusive, then the definite integral of $f(x)\,dx$ between a and b is defined as the limit, as n increases indefinitely, of the following sum of n terms,

$$f(a)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_{n-1})\Delta x,$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_1, x_2, x_3, \cdots, x_{n-1}$ are values of x between a and b such that

$$x_1 = a + \Delta x,\ \ x_2 = x_1 + \Delta x,\ \ x_3 = x_2 + \Delta x,\ \ \cdots,\ \ b = x_{n-1} + \Delta x.$$

The sum in the above definition is expressed concisely by the notation

$$\sum_{i=0}^{i=n-1} f(x_i)\Delta x,$$

where $\sum$ (sigma), the Greek form of the letter S, stands for the word "sum," and the whole expression indicates that the sum is to be taken of all terms obtained from $f(x_i)\Delta x$ by giving to i in succession the values 0, 1, 2, 3, $\cdots$, $n-1$, where $x_0 = a$.

Also the definite integral is denoted by the symbol

$$\int_a^b f(x)\,dx,$$

where $\int$ is a modified form of S. Hence the definition of the definite integral may be expressed symbolically by the equation

$$\int_a^b f(x)\,dx = \operatorname*{Lim}_{n=\infty} \sum_{i=0}^{i=n-1} f(x_i)\Delta x.$$

The numbers a and b are called the *lower* and the *upper limit*[*] respectively of the definite integral.

Ex. The conception of mechanical *work* gives an illustration of a definite integral. By definition, the work done in moving a body against a constant force is equal to the force multiplied by the distance through which the body is moved. Suppose now that a body is moved along OX (fig. 3) from A $(x = a)$ to B $(x = b)$ against a force which is not constant but a function of x and expressed by $f(x)$. Let the line AB be divided into n equal intervals, each equal to Δx, by the points M_1, M_2, M_3, $\cdots$, M_{n-1}. (In fig. 3, $n = 7$.)

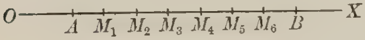

FIG. 3

Then the work done in moving the body from A to M_1 would be $f(a)\,\Delta x$ if the force were constantly equal to $f(a)$ throughout the interval AM_1. Consequently, if the interval is small, $f(a)\,\Delta x$ is approximately equal to the work done between A and M_1. Similarly, the work done between M_1 and M_2 is approximately equal to $f(x_1)\,\Delta x$, that between M_2 and M_3 approximately equal to $f(x_2)\,\Delta x$, and so on. Hence the work done between A and B is approximately equal to

$$f(a)\,\Delta x + f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_{n-1})\,\Delta x.$$

The larger the value of n, the better is this approximation. Hence we have, if W represents the work done between A and B,

$$W = \operatorname*{Lim}_{n=\infty} \sum_{i=0}^{i=n-1} f(x_i)\,\Delta x = \int_a^b f(x)\,dx.$$

The use of the word "integral" and of the symbol $\int$ suggests a connection with the integrals of the previous chapter. This connection will be shown in § 25; for the present, it is to be emphasized that the definition is independent of either differentiation or integration as previously known.

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[*] The student should notice that the word "limit" is here used in a sense quite different from that in which it is used when a variable is said to approach a limit (I, § 53).

In distinction to the definite integral $\int_a^b f(x)\,dx$, the integral $\int f(x)\,dx$ is called the *indefinite integral.*

The definition assumes that the limit of $\sum f(x_i)\Delta x$ always exists. A rigorous proof of this will not be given here, but the student will find it geometrically obvious from the graphical representation of the next article.

22. Graphical representation. Let LK (fig. 4) be the graph of $f(x)$, and let $OA = a$ and $OB = b$.

For convenience, we assume in the first place that $a < b$ and that $f(x)$ is positive for all values of x between a and b. We find $\Delta x = \dfrac{b-a}{n}$ and lay off the equal lengths $AM_1 = M_1M_2 = M_2M_3 = \cdots = M_{n-1}B = \Delta x$. (In fig. 4, $n = 9$.)

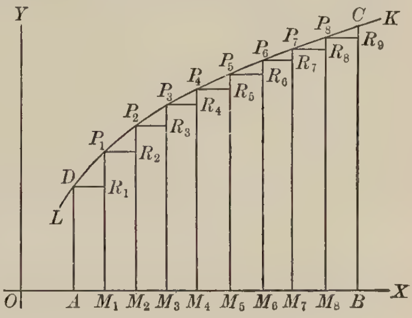

FIG. 4

Let $OM_1 = x_1$, $OM_2 = x_2$, $\cdots$, $OM_{n-1} = x_{n-1}$. Draw the ordinates $AD = f(a)$, $M_1P_1 = f(x_1)$, $M_2P_2 = f(x_2)$, $\cdots$, $M_{n-1}P_{n-1} = f(x_{n-1})$, and BC. Draw also the lines DR_1, P_1R_2, P_2R_3, $\cdots$, $P_{n-1}R_n$ parallel to OX. Then

$$f(a)\Delta x = \text{the area of the rectangle } ADR_1M_1,$$

$$f(x_1)\Delta x = \text{the area of the rectangle } M_1P_1R_2M_2,$$

$$f(x_2)\Delta x = \text{the area of the rectangle } M_2P_2R_3M_3,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$f(x_{n-1})\Delta x = \text{the area of the rectangle } M_{n-1}P_{n-1}R_nB.$$

The sum

$$f(a)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_{n-1})\Delta x$$

is then the sum of the areas of these rectangles, and equal to the area of the polygon $ADR_1P_1R_2 \cdots R_{n-1}P_{n-1}R_nB$. It is evident that the limit of this sum as n is indefinitely increased is the area bounded by AD, AB, BC, and the arc DC.

Hence $$\int_a^b f(x)\,dx = \text{the area } ABCD.$$

It is evident that the result is not vitiated if AD or BC is of length zero.

The area $ABCD$ is exactly the sum of the areas ADP_1M_1, $M_1P_1P_2M_2$, $M_2P_2P_3M_3$, etc. But one such area, for example $M_1P_1P_2M_2$, differs from that of the corresponding rectangle $M_1P_1R_2M_2$ by the area of the figure $P_1R_2P_2$, which is less than that of the rectangle $\Delta x\,\Delta y$, where $\Delta y = R_2P_2$. The area of $P_1R_2P_2$ is an infinitesimal of higher order than $M_1P_1R_2M_2$, since $\frac{P_1R_2P_2}{M_1P_1R_2M_2} < \frac{\Delta x\,\Delta y}{y\,\Delta x} = \frac{\Delta y}{y}$, whence $\text{Lim}\,\frac{P_1R_2P_2}{M_1P_1R_2M_2} = 0$. Therefore (§ 3) the areas of the triangular figures such as $P_1R_2P_2$ do not affect the limit of the sum used in finding the area of the entire figure.

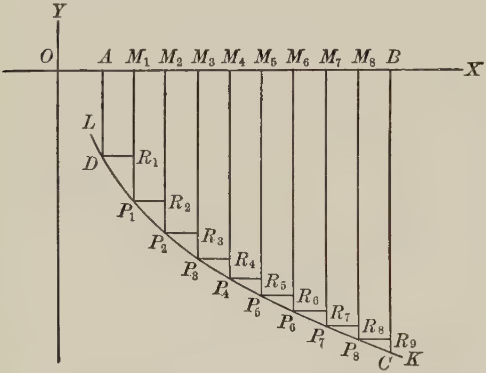

FIG. 5

If $f(x)$ is negative for all values of x between a and $b\,(a<b)$, the graphical representation is as in fig. 5. Here

$$f(a)\,\Delta x = -\text{ the area of the rectangle } AM_1R_1D,$$

$$f(x_1)\,\Delta x = -\text{ the area of the rectangle } M_1M_2R_2P_1, \text{ etc.,}$$

so that $$\int_a^b f(x)\,dx = -\text{ the area } ABCD.$$

In case $f(x)$ is sometimes positive and sometimes negative we have a combination of the foregoing results, as follows:

If $a<b$, *the integral* $\int_a^b f(x)\,dx$ *represents the algebraic sum of the areas bounded by the curve* $y=f(x)$, *the axis of* x, *and the ordinates* $x=a$ *and* $x=b$, *the areas above the axis of* x *being positive and those below negative.*

If $a > b$, Δx is negative, since $\Delta x = \frac{b-a}{n}\cdot$ The only change necessary in the above statement, however, is in the algebraic signs, the areas above the axis of x being now negative and those below positive.

Ex. The work done in moving a body against a force may be represented by an area. For if the force is $f(x)$ and W is the work in moving a body along the axis of x from $x = a$ to $x = b$, then (Ex., § 21)

$$W = \int_a^b f(x)\,dx.$$

Consequently if the force is represented by the graph DEC (fig. 6), the equation of which is $y = f(x)$, the area $ABCED$ represents the work W.

This fact is taken advantage of in constructing an indicator diagram attached to a steam engine. Here AB represents the distance traversed by the piston, and the ordinate represents the pressure. Then as the piston travels from A to B and back to A the curve $DECFD$ is automatically drawn. The area $ABCED$ represents the work done by the steam on the piston. The area $ABCFD$ represents the work done by the piston on the steam. The difference of these two areas, which is the area of the closed curve $DECFD$, represents the net work done by the steam in one stroke of the piston. In practice this area can be measured by an instrument, called a planimeter, or the figure is divided into rectangles and the area computed approximately. The latter method illustrates the definition of the definite integral.

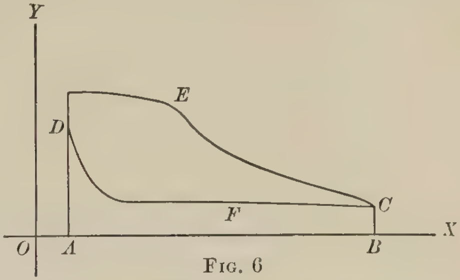

FIG. 6

23. Generalization. In the definition of § 21,

$$x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \cdots = x_n - x_{n-1} = \Delta x,$$

where $x_0 = a$ and $x_n = b$. The sum

$$f(a)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_{n-1})\Delta x$$

may accordingly be written

$$f(x_0)(x_1 - x_0) + f(x_1)(x_2 - x_1) \\ + f(x_2)(x_3 - x_2) + \cdots + f(x_{n-1})(x_n - x_{n-1}),$$

or, more compactly,

$$\sum_{i=0}^{i=n-1} f(x_i)(x_{i+1} - x_i). \qquad (1)$$

This sum may be generalized in two ways:

In the first place, it is a mere matter of convenience to take the increments $x_{i+1}-x_i$ equal for all values of x_i. In fact the $n-1$ values $x_1, x_2, x_3, \cdots, x_{n-1}$ may be taken at pleasure between the values $x=a$ and $x=b$ without altering the limit approached by the sum (1), provided all the differences $x_{i+1}-x_i$ are made to approach zero as n increases without limit. This is geometrically obvious from the graphical representation, for the bases AM_1, M_1M_2, $M_2M_3, \cdots$ of the rectangles of fig. 4 may be of unequal length.

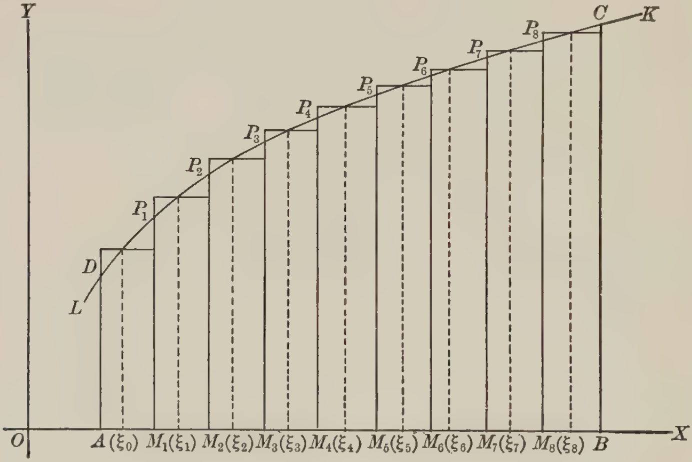

FIG. 7

In the second place, the factor $f(x_i)$ in each of the terms of the sum (1) may be replaced by $f(\xi_i)$, where ξ_i is any value of x between x_i and x_{i+1}. The effect on the graphical representation of fig. 4 is to alter the altitudes of the rectangles without altering the limit of their sum, as exemplified in fig. 7. It may be noted that the rectangles here differ from those of fig. 4 by infinitesimals of higher order. Hence the sum theorem of § 3 applies.

For a rigorous discussion of these points the student is referred to advanced treatises.*

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* See, for example, Goursat-Hedrick, *Mathematical Analysis*, Chap. IV.

24. Properties of definite integrals. The following properties of definite integrals are consequences of the definition:

1. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx,$
2. $\int_a^b [f_1(x)+f_2(x)]\,dx = \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx,$
3. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx,$
4. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$
5. $\int_a^b f(x)\,dx = (b-a)f(\xi),$ where $a < \xi < b.$

The truth of formulas 1 and 2 follows at once from the definition, and that of formula 3 follows at once from § 22. We shall show the truth of formulas 4 and 5 graphically. A fully rigorous proof could be based on the definition of § 21 without the use of diagrams, but would follow the outlines of the following graphical discussion.

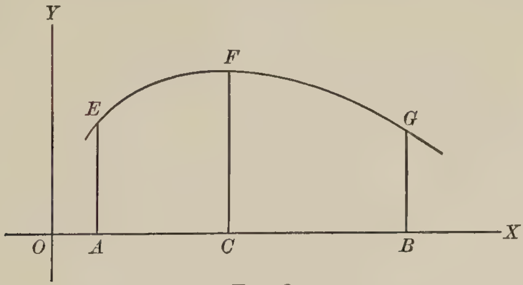

FIG. 8

To prove formula 4, consider fig. 8, where $OA = a$, $OC = c$, $OB = b$. Then $\int_a^c f(x)\,dx =$ the area $ACFE$, $\int_c^b f(x)\,dx =$ the area $CBGF$, and $\int_a^b f(x)\,dx =$ the area $ABGE$. The truth of formula 4 is apparent for any order of the points A, C, B, reference being had, if necessary, to formula 3.

To prove formula 5, consider fig. 9, where the area *ABCD* represents the value of $\int_a^b f(x)\,dx$, and let m and M respectively be the smallest and the largest value assumed by $f(x)$ in the interval *AB*. Construct the rectangle *ABKH* with the base *AB* and the altitude $AH = M$. Its area is $AB \cdot AH = (b-a)M$. Construct also the rectangle *ABLN* with the base *AB* and the altitude $AN = m$. Its area is $AB \cdot AN = (b-a)m$.

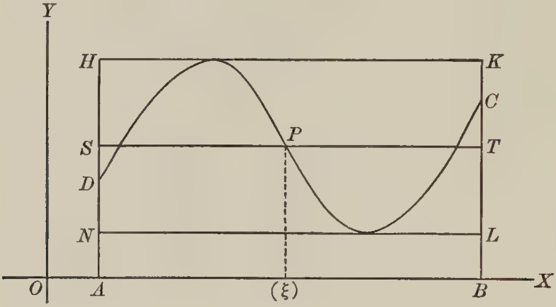

FIG. 9

Now it is evident that the area *ABCD* is greater than the area *ABLN* and less than the area *ABKH*. That is,

$$(b-a)m < \int_a^b f(x)\,dx < (b-a)M.^*$$

Consequently $$\int_a^b f(x)\,dx = (b-a)\mu,$$

where μ is some quantity greater than m and less than M, and is represented on fig. 9 by *AS*. But since $f(x)$ is a continuous function, there is at least one value ξ between a and b such that $f(\xi) = \mu$, and therefore

$$\int_a^b f(x)\,dx = (b-a)f(\xi).$$

Graphically, this says that the area *ABCD* is equal to a rectangle *ABTS* whose base is *AB* and whose altitude *AS* lies between *AN* and *AH*.

---

* A slight modification is here necessary if $f(x) = k$, a constant. Then $M = m = k$ and $\int_a^b f(x)\,dx = (b-a)k$.

25. Evaluation of the definite integral by integration. When $f(x)$ is a known function and a and b are constants, the value of the integral is fully determined. Hence if we replace the upper limit b by a variable x, the definite integral is a function of x. Graphically (fig. 10), $\int_a^x f(x)\,dx =$ area $AMPC$, where $OA = a$, a constant, and $OM = x$, a variable. Let us place

$$\phi(x) = \int_a^x f(x)\,dx = \text{the area } AMPC.$$

Now we have shown in I, § 109, 6, that

$$\frac{d}{dx}\phi(x) = y = f(x).$$

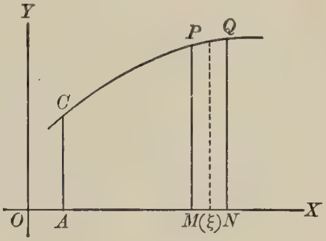

FIG. 10

A new proof of this can be given by use of the properties of § 24, and this proof has the advantage of being really independent of the graphical representation, although for convenience we shall refer to the figure.

Take $MN = h$. Then

$$\phi(x+h) = \int_a^{x+h} f(x)\,dx = \text{the area } ANQC,$$

whence $$\phi(x+h) - \phi(x) = \int_x^{x+h} f(x)\,dx$$

$$= \text{the area } MNQP, \quad \text{(by 3 and 4, § 24)}$$

$$= hf(\xi), \quad \text{(by 5, § 24)}$$

where $x < \xi < x + h$. Therefore

$$\frac{\phi(x+h) - \phi(x)}{h} = f(\xi).$$

Taking now the limit as h approaches zero, and remembering that

$\operatorname*{Lim}_{h \doteq 0} \frac{\phi(x+h) - \phi(x)}{h} = \frac{d}{dx}\phi(x)$ and $\operatorname*{Lim}_{h \doteq 0} \xi = x$, we have

$$\frac{d}{dx}\phi(x) = f(x).$$

Let now $F(x)$ be any known function whose derivative is $f(x)$. Then $F(x)$ differs from $\phi(x)$ by a constant (§ 30), that is,

$$\int_a^x f(x)\,dx = F(x) + C.$$

To determine C, place $x = a$, and notice that $\int_a^a f(x)\,dx = 0$.

Then $\quad 0 = F(a) + C, \quad$ whence $\quad C = -F(a)$.

Therefore
$$\int_a^x f(x)\,dx = F(x) - F(a),$$

whence
$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This result gives the following rule for evaluating a definite integral:

To find the value of $\int_a^b f(x)\,dx$, *evaluate* $\int f(x)\,dx$, *substitute* $x = b$ *and* $x = a$ *successively, and subtract the latter result from the former.*

It is to be noticed that in evaluating $\int f(x)\,dx$ the constant of integration is to be omitted, since $-F(a)$ is that constant. However, if the constant is added, it disappears in the subtraction, since

$$[F(b) + C] - [F(a) + C] = F(b) - F(a).$$

In practice it is convenient to express $F(b) - F(a)$ by the symbol $[F(x)]_a^b$, so that

$$\int_a^b f(x)\,dx = [F(x)]_a^b.$$

Ex. 1. $\int_1^3 x^3\,dx = \left[\frac{x^4}{4}\right]_1^3 = \frac{81}{4} - \frac{1}{4} = 20.$

Ex. 2. $\int_0^{\frac{\pi}{2}} \sin x\,dx = [-\cos x]_0^{\frac{\pi}{2}} = -\cos\frac{\pi}{2} + \cos 0 = 1.$

Ex. 3. $\int_0^1 \frac{4x+2}{x^2+x+2}\,dx = [2\log(x^2+x+2)]_0^1 = 2\log 4 - 2\log 2 = \log 4.$

Ex. 4. $\int_{-1}^{\sqrt{3}} \frac{dx}{1+x^2} = [\tan^{-1}x]_{-1}^{\sqrt{3}} = \tan^{-1}\sqrt{3} - \tan^{-1}(-1).$

There is here a certain ambiguity, since $\tan^{-1}\sqrt{3}$ and $\tan^{-1}(-1)$ have each an infinite number of values. If, however, we remember that the graph of $\tan^{-1}x$ is composed of an infinite number of distinct parts, or *branches*, the ambiguity is removed by taking the values of $\tan^{-1}\sqrt{3}$ and $\tan^{-1}(-1)$ from the same branch of the function. For if we consider $\int_a^b \frac{dx}{1+x^2} = \tan^{-1}b - \tan^{-1}a$ and select any value of $\tan^{-1}a$, then if $b = a$, $\tan^{-1}b$ must be taken equal to $\tan^{-1}a$, since the value of the integral is then zero. As b varies from equality with a to its final value, $\tan^{-1}b$ will vary from $\tan^{-1}a$ to the nearest value of $\tan^{-1}b$.

The simplest way to choose the proper values of $\tan^{-1}b$ and $\tan^{-1}a$ is to take them both between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Then we have

$$\int_{-1}^{\sqrt{3}} \frac{dx}{1+x^2} = \frac{\pi}{3} - \left(-\frac{\pi}{4}\right) = \frac{7\pi}{12}.$$

Ex. 5. $\int_0^{\frac{a}{2}} \frac{dx}{\sqrt{a^2-x^2}} = \left[\sin^{-1}\frac{x}{a}\right]_0^{\frac{a}{2}} = \sin^{-1}\frac{1}{2} - \sin^{-1}0.$

The ambiguity in the values of $\sin^{-1}\frac{1}{2}$ and $\sin^{-1}0$ is removed by noticing that $\sin^{-1}\frac{x}{a}$ must lie in the fourth or the first quadrant, and that the two values must be so chosen that one comes out of the other by continuous change. The simplest way to accomplish this is to take both $\sin^{-1}\frac{1}{2}$ and $\sin^{-1}0$ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Then
$$\int_0^{\frac{a}{2}} \frac{dx}{\sqrt{a^2-x^2}} = \frac{\pi}{6} - 0 = \frac{\pi}{6}.$$

26. Change of limits. In case it is convenient to evaluate $\int f(x)\,dx$ by substitution, the limits of $\int_a^b f(x)\,dx$ may be replaced by the corresponding values of the variable substituted. To see this, suppose that in $\int f(x)\,dx$ the variable x is replaced by a function of a new variable t, such that when x varies continuously from a to b, t varies continuously from t_0 to t_1. Let the work of finding the indefinite integral be indicated as follows:

$$\int f(x)\,dx = \int \phi(t)\,dt = \Phi(t) = F(x),$$

where $F(x)$ is obtained by replacing t in $\Phi(t)$ by its value in terms of x. Then

$$F(b) - F(a) = \Phi(t_1) - \Phi(t_0).$$

But
$$F(b)-F(a)=\int_a^b f(x)\,dx$$

and
$$\Phi(t_1)-\Phi(t_0)=\int_{t_0}^{t_1}\phi(t)\,dt.$$

Hence
$$\int_a^b f(x)\,dx=\int_{t_0}^{t_1}\phi(t)\,dt.$$

Ex. 1. Find $\int_0^a \sqrt{a^2-x^2}\,dx$.

Place $x=a\sin\phi$. Then $dx=a\cos\phi\,d\phi$, and when x varies from 0 to a, ϕ varies from 0 to $\frac{\pi}{2}$. Hence

$$\int_0^a \sqrt{a^2-x^2}\,dx=\int_0^{\frac{\pi}{2}} a^2\cos^2\phi\,d\phi=\left[\frac{a^2\phi}{2}+\frac{a^2\sin 2\phi}{4}\right]_0^{\frac{\pi}{2}}=\frac{1}{4}\pi a^2.$$

In making the substitution care should be taken that to each value of x between a and b corresponds one and only one value of t between t_0 and t_1, and conversely. Failure to do this may lead to error.

Ex. 2. Consider $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\phi\,d\phi$, which by direct integration is equal to 2.

Let us place $\cos\phi=x$, whence $\phi=\cos^{-1}x$ and $d\phi=\frac{\mp\,dx}{\sqrt{1-x^2}}$, where the sign depends upon the quadrant in which ϕ is found. We cannot, therefore, make this substitution in $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\phi\,d\phi$, since ϕ lies in two different quadrants; but we may write

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\phi\,d\phi=\int_{-\frac{\pi}{2}}^{0}\cos\phi\,d\phi+\int_0^{\frac{\pi}{2}}\cos\phi\,d\phi,$$

and in the first of the integrals on the right-hand side of this equation place $\phi=\cos^{-1}x$, $d\phi=\frac{dx}{\sqrt{1-x^2}}$, and in the second $\phi=\cos^{-1}x$, $d\phi=\frac{-\,dx}{\sqrt{1-x^2}}$. Then

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\phi\,d\phi=\int_0^1\frac{x\,dx}{\sqrt{1-x^2}}-\int_1^0\frac{x\,dx}{\sqrt{1-x^2}}=2\int_0^1\frac{x\,dx}{\sqrt{1-x^2}}=2.$$

Ex. 3. Consider $\int_{-1}^{+1}dx$ and place $x^2=t$.

Then, when $x=-1$, $t=1$, and when $x=1$, $t=1$; and the attempt to substitute without care would lead to error. But $x=-t^{\frac{1}{2}}$ and $dx=-\frac{1}{2}t^{-\frac{1}{2}}dt$ when x is negative; and $x=t^{\frac{1}{2}}$ and $dx=\frac{1}{2}t^{-\frac{1}{2}}dt$ when x is positive. Hence

$$\int_{-1}^{+1}dx=\int_{-1}^{0}dx+\int_0^1 dx=-\int_1^0\tfrac{1}{2}t^{-\frac{1}{2}}dt+\int_0^1\tfrac{1}{2}t^{-\frac{1}{2}}dt=\int_0^1 t^{-\frac{1}{2}}dt=2.$$

27. Integration by parts. If it is desired to integrate by parts, and a and b are values of the independent variable, then

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du.$$

To prove this, note that it follows at once from the equation

$$[uv]_a^b = \int_a^b d(uv) = \int_a^b (u\,dv + v\,du) = \int_a^b u\,dv + \int_a^b v\,du.$$

Ex. 1. Find $\int_0^1 xe^x dx$.

Take $x = u$, $e^x dx = dv$; then

$$\int_0^1 xe^x dx = [xe^x]_0^1 - \int_0^1 e^x dx = e - [e^x]_0^1 = 1.$$

Ex. 2. Discuss $\int_0^{\frac{\pi}{2}} \sin^n x\,dx$.

Take $\sin^{n-1}x = u$, $\sin x\,dx = dv$; then

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^n x\,dx &= [-\cos x \sin^{n-1}x]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}x \cos^2 x\,dx\\ &= (n-1)\int_0^{\frac{\pi}{2}} (\sin^{n-2}x - \sin^n x)\,dx.\end{aligned}$$

By transposing we have

$$n\int_0^{\frac{\pi}{2}} \sin^n x\,dx = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}x\,dx,$$

whence

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2}x\,dx. \qquad (1)$$

If, in (1), we place $n = 2$, we have

$$\int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} dx = \frac{1}{2}\cdot\frac{\pi}{2}.$$

If, in (1), we place $n = 3$, we have

$$\int_0^{\frac{\pi}{2}} \sin^3 x\,dx = \frac{2}{3}\int_0^{\frac{\pi}{2}} \sin x\,dx = \frac{2}{3}.$$

If, in (1), we place $n = 4$, we have

$$\int_0^{\frac{\pi}{2}} \sin^4 x\,dx = \frac{3}{4}\int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2}.$$

If, in (1), we place $n = 5$, we have

$$\int_0^{\frac{\pi}{2}} \sin^5 x\, dx = \frac{4}{5}\int_0^{\frac{\pi}{2}} \sin^3 x\, dx = \frac{4 \cdot 2}{5 \cdot 3}.$$

Continuing in this way, we find

$$\int_0^{\frac{\pi}{2}} \sin^{2k} x\, dx = \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2 \cdot 4 \cdot 6 \cdots 2k} \cdot \frac{\pi}{2},$$

$$\int_0^{\frac{\pi}{2}} \sin^{2k+1} x\, dx = \frac{2 \cdot 4 \cdot 6 \cdots 2k}{3 \cdot 5 \cdot 7 \cdots (2k+1)}.$$

28. Infinite limits. It is possible to have the upper limit ∞, where by definition

$$\int_a^\infty f(x)\, dx = \operatorname*{Lim}_{b=\infty} \int_a^b f(x)\, dx.$$

The integral, then, is represented by the area bounded by the curve $y = f(x)$, the axis of x, and the ordinate $x = a$, the figure being unbounded at the right hand. There is no certainty that such an area is either finite or determinate. The tests by which it may be sometimes determined whether $\int_a^\infty f(x)\, dx$ has a meaning will not be given. In case, however, it is possible to find the indefinite integral $F(x) = \int f(x)\, dx$, the definite integral can be found by the formula

$$\int_a^\infty f(x)\, dx = F(\infty) - F(a),$$

where $$F(\infty) = \operatorname*{Lim}_{b=\infty} F(b).$$

Ex. 1. $\int_1^\infty \frac{dx}{\sqrt{x}} = \left[2\sqrt{x}\right]_1^\infty = \infty.$ (fig. 11)

Ex. 2. $\int_1^\infty \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_1^\infty = 1.$ (fig. 12)

Ex. 3. $\int_0^\infty \sin x\, dx = [-\cos x]_0^\infty$
= indeterminate. (fig. 13)

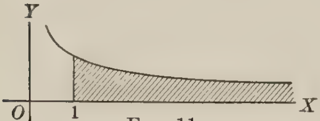

FIG. 11

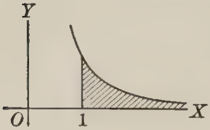

FIG. 12

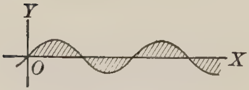

FIG. 13

Similarly, the lower limit, or both limits, of the definite integral may be infinite.

29. Infinite integrand. According to the definition of § 21 it is unallowable that $f(x)$ in $\int_a^b f(x)\,dx$ should become infinite between $x=a$ and $x=b$. It is, however, possible to admit the case in which $f(x)$ becomes infinite when $x=b$ by means of the definition

$$\int_a^b f(x)\,dx = \operatorname*{Lim}_{h \doteq 0} \int_a^{b-h} f(x)\,dx,$$

and to compute the integral by the formula

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F(b)$ means $\operatorname*{Lim}_{h \doteq 0} F(b-h)$.

The integral has not, however, necessarily a finite or determinate value. Graphically, the integral is represented by the area between the curve $y=f(x)$, the axis of x, the ordinate $x=a$, and the asymptote $x=b$.

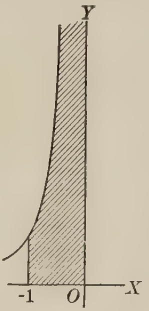

FIG. 14

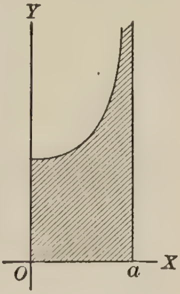

FIG. 15

Ex. 1. $\int_{-1}^{0} \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_{-1}^{0} = \infty.$ (fig. 14)

Ex. 2. $\int_0^a \frac{dx}{\sqrt{a^2-x^2}} = \left[\sin^{-1}\frac{x}{a}\right]_0^a = 1.$ (fig. 15)

Similarly, $f(x)$ may become infinite at the lower limit or at both limits. If it becomes infinite for any value c between the limits, the integral should be separated into two integrals having c for the upper and the lower limit respectively. Failure to do this may lead to error.

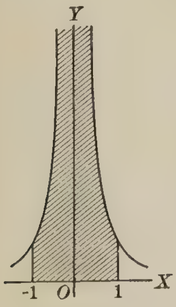

FIG. 16

Ex. 3. Consider $\int_{-1}^{+1} \frac{dx}{x^2}$.

Since $\frac{1}{x^2}$ becomes infinite when $x=0$ (fig. 16), we separate the integral into two, thus:

$$\int_{-1}^{+1} \frac{dx}{x^2} = \int_{-1}^{0} \frac{dx}{x^2} + \int_0^1 \frac{dx}{x^2} = \infty$$

Had we carelessly applied the incorrect formula

$$\int_{-1}^{+1}\frac{dx}{x^2}=\left[-\frac{1}{x}\right]_{-1}^{+1},$$

we should have been led to the absurd result -2.

30. The mean value of a function. We have seen in § 24 that

$$f(\xi)=\frac{1}{b-a}\int_a^b f(x)\,dx, \tag{1}$$

where ξ lies between a and b. The value

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

is called the *mean value* of $f(x)$ in the interval from a to b. This is in fact an extension of the ordinary meaning of the average, or mean, value of n measurements. For let $y_0, y_1, y_2, \cdots, y_{n-1}$, correspond to n values of x, which divide the interval from a to b into n equal parts, each equal to Δx. Then the average of these n values of y is

$$\frac{y_0+y_1+y_2+\cdots+y_{n-1}}{n}.$$

This fraction is equal to

$$\frac{(y_0+y_1+y_2+\cdots+y_{n-1})\,\Delta x}{n\Delta x}=\frac{y_0\Delta x+y_1\Delta x+y_2\Delta x+\cdots+y_{n-1}\Delta x}{b-a}.$$

As n is indefinitely increased, this expression approaches as a limit $\frac{1}{b-a}\int_a^b y\,dx=\frac{1}{b-a}\int_a^b f(x)\,dx$. Hence the mean value of a function may be considered as the average of an "infinite number" of values of the function, taken at equal distances between a and b.

Ex. 1. Find the mean velocity of a body falling from rest during the time t_1. The velocity is gt, where g is the acceleration due to gravity. Hence the mean velocity is $\frac{1}{t_1-0}\int_0^{t_1} gt\,dt=\frac{1}{2}gt_1$. This is half the final velocity.

Ex. 2. In using the indicator diagram (§ 22) engineers use the "mean effective pressure," which is defined as the constant pressure which will do the same amount of work per stroke as is done by the varying pressure shown by the indicator diagram. It is found by dividing the area of the diagram by $b-a$, and is accordingly an example of the mean value of a function.

Formula (1) may be written in another form, not involving the integral sign. Let us place $\int f(x)\,dx = F(x)$; then $f(x) = F'(x)$, and (1) becomes

$$F'(\xi) = \frac{1}{b-a}[F(b) - F(a)], \tag{2}$$

or

$$F(b) - F(a) = (b-a)\,F'(\xi). \tag{3}$$

Formula (2) has a simple graphical meaning. For let LK (fig. 17) be the graph of $F(x)$ and let $OA = a$ and $OB = b$. Then $b - a = AB$, $F(b) - F(a) = BE - AD = CE$, and

$$\frac{F(b)-F(a)}{b-a} = \frac{CE}{DC} = \text{the slope of the chord } DE.$$

If now ξ is any value of x, $F'(\xi)$ is the slope of the tangent at the corresponding point of LK. Hence (2) asserts that there is some point between A and B for which the tangent is parallel to the chord DE. This is evidently true if $F(x)$ and $F'(x)$ are continuous.

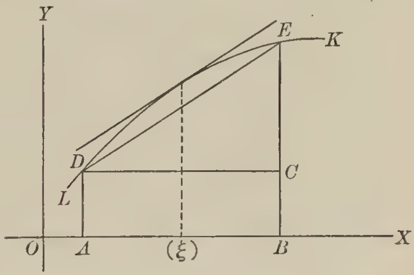

FIG. 17

Formula (3) may be used to prove the proposition, which we have previously used without proof, namely: *If the derivative of a function is always zero, the function is a constant.* For let $F'(x)$ be always zero and let a and b be any two values of x. Then, by (3), $F(b) - F(a) = 0$. That is, any two values of the function are equal; in other words, the function is a constant.

From this it follows that *two functions which have the same derivative differ by a constant.* For if $F'(x) = \Phi'(x)$, then $\frac{d}{dx}[F(x) - \Phi(x)] = 0$; whence $F(x) = \Phi(x) + C$.

31. Taylor's and Maclaurin's series. Formula (3), § 30, is a special case of a more general relation, which we will now proceed to obtain. Let us take the equation

$$\int_a^x f''(x)\,dx = f'(x) - f'(a),$$

multiply both sides by dx, and integrate between a and x. We have

$$\int_a^x\int_a^x f''(x)\,dx^2 = \int_a^x f'(x)\,dx - \int_a^x f'(a)\,dx$$
$$= f(x) - f(a) - (x-a)f'(a).$$

Therefore $$f(x) = f(a) + (x-a)f'(a) + \int_a^x\int_a^x f''(x)\,dx^2.$$

But if m is the smallest and M the largest value of $f''(x)$ between a and x, then (§ 24), when $x > a$,

$$m(x-a) < \int_a^x f''(x)\,dx < M(x-a),$$

whence $$\frac{m(x-a)^2}{2} < \int_a^x\int_a^x f''(x)\,dx^2 < \frac{M(x-a)^2}{2}.$$

Hence $$\int_a^x\int_a^x f''(x)\,dx^2 = \frac{\mu(x-a)^2}{2}, \quad \text{where} \quad m < \mu < M.$$

If $f''(x)$ is a continuous function, there is at least one value of x, say ξ_1, between a and x, for which $f''(\xi_1) = \mu$ (I, § 56). Therefore we have, finally,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(\xi_1). \qquad (1)$$

Again, let us take

$$\int_a^x f'''(x)\,dx = f''(x) - f''(a).$$

Then $$\int_a^x\int_a^x f'''(x)\,dx^2 = \int_a^x f''(x)\,dx - \int_a^x f''(a)\,dx$$
$$= f'(x) - f'(a) - (x-a)f''(a),$$

and

$$\int_a^x\int_a^x\int_a^x f'''(x)\,dx^3 = \int_a^x f'(x)\,dx - \int_a^x f'(a)\,dx - \int_a^x (x-a)f''(a)\,dx$$
$$= f(x) - f(a) - (x-a)f'(a) - \frac{(x-a)^2}{2}f''(a),$$

whence

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{\underline{|2}}f''(a) + \frac{(x-a)^3}{\underline{|3}}f'''(\xi_2)^* \qquad (2)$$

where $a < \xi_2 < x$.

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* The symbol $\underline{|2}$ means $1\cdot 2$, $\underline{|3}$ means $1\cdot 2\cdot 3$, and, in general, $\underline{|n}$, read factorial n, means the product of the n integers from 1 to n inclusive.

Similarly, if we start with $\int_a^x f^{(n+1)}(x)\,dx$, we obtain

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{\underline{|2}} f''(a) + \frac{(x-a)^3}{\underline{|3}} f'''(a) + \cdots + \frac{(x-a)^n}{\underline{|n}} f^n(a) + R_n, \tag{3}$$

where $R_n = \dfrac{(x-a)^{n+1}}{\underline{|n+1}} f^{(n+1)}(\xi)$, where $a < \xi < x$, the only restrictions being that the $n+1$ derivatives of $f(x)$ exist and are continuous.

The value of R_n in (3) is not known exactly, since ξ is unknown, but it usually happens that, if $x - a$ is a sufficiently small quantity, $\lim\limits_{n=\infty} R_n = 0$. Then the value of $f(x)$ may be expressed approximately by the first $n+1$ terms of (3), omitting R_n, and this approximation approaches $f(x)$ as a limit as n is indefinitely increased. In this case we say that $f(x)$ is expanded into the *infinite series*

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{\underline{|2}} f''(a) + \frac{(x-a)^3}{\underline{|3}} f'''(a) + \cdots. \tag{4}$$

For larger values of $x - a$, however, it may happen that the value of R_n increases without limit as n increases. Hence the omission of R_n in (3) leaves $n+1$ terms, the sum of which differs more and more from $f(x)$ the more terms are taken. In such a case the series (4) cannot be taken to represent the function.

The determination of the values of $x - a$ for which (4) is valid is, in general, a matter involving a knowledge of mathematics which lies outside the limits of this course. In the illustrative examples and in the problems for the student we shall simply state the facts in each case without proof.

Formula (4) is known as *Taylor's series*. Here a is a known value of x for which $f(x)$ and its derivatives are known. The series then enables us to compute the value of $f(x)$ for values of x not too remote from a.

Another convenient form of (4) is obtained by placing $x - a = h$, whence $x = a + h$. We have, then,

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2}f''(a) + \frac{h^3}{\lfloor 3}f'''(a) + \cdots. \tag{5}$$

Here the remainder is usually expressed as

$$R_n = \frac{h^{n+1}}{\lfloor n+1}f^{n+1}(a+\theta h), \quad \text{where} \quad 0 < \theta < 1.$$

A special form of (4) arises when $a = 0$. We have, then,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2}f''(0) + \frac{x^3}{\lfloor 3}f'''(0) + \cdots. \tag{6}$$

This is known as *Maclaurin's series*, and the function is said to be expanded into a *power series in x*.

Ex. 1. e^x.

By Maclaurin's series (6), we find, since $f(x) = e^x$, $f'(x) = e^x$, $f''(x) = e^x$, etc., and $f(0) = 1$, $f'(0) = 1$, $f''(0) = 1$, etc.,

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \cdots.$$

This expansion is valid for all values of x. If we place $x = \frac{1}{3}$, we have $e^{\frac{1}{3}} = 1 + \frac{1}{3} + \frac{1}{18} + \frac{1}{162} + \frac{1}{1944} = 1.3956$, correct to four decimal places. If x has a larger value, more terms of the series must be taken in the computation, so that the series, while valid, is inconvenient for large values of x.

Ex. 2. $\sin x$.

By Maclaurin's series,

$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \cdots,$$

which is valid for all values of x.

To find $\sin 15°$, we first change 15° to circular measure, which is $\frac{15}{180}\pi = \frac{1}{12}\pi = .2618$. Then the first two terms of the series give $\sin 15° = .2588$, correct to four decimal places.

By Taylor's series (5), we have

$$\sin(a+h) = \sin a + h\cos a - \frac{h^2}{\lfloor 2}\sin a - \frac{h^3}{\lfloor 3}\cos a + \cdots.$$

Ex. 3. $\cos x$.

By Maclaurin's series,

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots.$$

Ex. 4. $(a+x)^n$.

By Maclaurin's series,

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{\underline{|2}}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{\underline{|3}}a^{n-3}x^3 + \cdots.$$

This is the *binomial theorem.* If n is a positive integer, the expansion is a polynomial of $n+1$ terms, since $f^{(n+1)}(x)$ and all higher derivatives are equal to 0. But if n is a negative integer or a fraction, the expansion is an infinite series which is valid when x is numerically less than a.

Ex. 5. $\log x$.

The function $\log x$ cannot be expanded by Maclaurin's series, since its derivatives are infinite when $x=0$. We may use Taylor's series (4), placing $a=1$. We have, then,

$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots.$$

Or we may expand $\log(1+x)$ by Maclaurin's series with the result

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

This expansion is valid when x is numerically less than 1.

32. Operations with power series. When a function is expressed as a power series, it may be integrated or differentiated by integrating or differentiating the series term by term. The new series will be valid for the same values of the variable for which the original series is valid. If the method is applied to a definite integral, the limits must be values for which the series is valid.

Similarly, if two functions are each expressed by a power series, their sum, difference, product, or quotient is the sum, the difference, the product, or the quotient of the series.

Ex. 1. Required to expand $\sin^{-1}x$.

We have

$$\sin^{-1}x = \int_0^x \frac{dx}{\sqrt{1-x^2}} = \int_0^x (1-x^2)^{-\frac{1}{2}}dx$$

$$= \int_0^x \left(1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \cdots\right)dx \quad \text{(by Ex. 4, § 31)}$$

$$= x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7} + \cdots.$$

Ex. 2. To find $\int_0^x e^{-x^2}dx$.

The indefinite integral cannot be found directly. We may expand e^{-x^2} by Ex. 1, § 31. Then

$$\int_0^x e^{-x^2}dx = \int_0^x\left(1-\frac{x^2}{1}+\frac{x^4}{\underline{|2}}-\frac{x^6}{\underline{|3}}+\cdots\right)dx = x-\frac{x^3}{3}+\frac{x^5}{5\underline{|2}}-\frac{x^7}{7\underline{|3}}+\cdots.$$

Ex. 3. To expand $\dfrac{1}{(1+x)^3}$.

By division, $$\frac{1}{1+x}=1-x+x^2-x^3+x^4-x^5+\cdots.$$

By successive differentiation,

$$-(1+x)^{-2}=-1+2x-3x^2+4x^3-5x^4+\cdots,$$
$$2(1+x)^{-3}=2-6x+12x^2-20x^3+\cdots.$$

Therefore $$(1+x)^{-3}=1-3x+6x^2-10x^3+\cdots.$$

Ex. 4. To expand $\log\dfrac{1+x}{1-x}$.

$$\begin{aligned}\log\frac{1+x}{1-x} &= \log(1+x)-\log(1-x)\\ &=\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots\right)-\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}-\cdots\right)\\ &=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots\right).\end{aligned}$$

Ex. 5. To expand $\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$.

By Ex. 1, $$\sin^{-1}x=x+\frac{x^3}{6}+\frac{3x^5}{40}+\frac{5x^7}{112}+\cdots;$$

by Ex. 4, § 31, $$(1-x^2)^{-\frac{1}{2}}=1+\frac{x^2}{2}+\frac{3x^4}{8}+\frac{5x^6}{16}+\cdots.$$

Hence, by multiplication,

$$\frac{\sin^{-1}x}{\sqrt{1-x^2}}=x+\frac{2x^3}{3}+\frac{8x^5}{15}+\frac{16x^7}{35}+\cdots.$$

33. By means of Taylor's theorem we can complete the rule given in I, § 62, for the maximum and the minimum values of a function of one variable. Let a be a value of x for which the first n derivatives of $f(x)$ are zero; i.e. let $f'(a)=0$, $f''(a)=0$, $f'''(a)=0$, $\cdots$, $f^{(n)}(a)=0$, but $f^{(n+1)}(a)\neq 0$. Then, by Taylor's theorem (5), § 31,

$$f(a+h)-f(a)=\frac{h^{n+1}f^{(n+1)}(a)}{\underline{|n+1}}+R_{n+1}.$$

If h is sufficiently small, the term $\dfrac{h^{n+1}f^{(n+1)}(a)}{\underline{|n+1}}$ will be larger numerically than R_{n+1}, since the latter contains h^{n+2} as a factor. Hence the sign of $f(a+h)-f(a)$ depends upon the sign of $\dfrac{h^{n+1}f^{(n+1)}(a)}{\underline{|n+1}}$.

If n is even, $n+1$ is odd, and the sign of h^{n+1} changes with the sign of h. Hence the sign of $f(a+h)-f(a)$ changes with that of h. Therefore $f(a)$ is neither a maximum nor a minimum value of $f(x)$.

If n is odd, $n+1$ is even and h^{n+1} is always positive. Hence the sign of $f(a \pm h)-f(a)$ is the same as the sign of $f^{(n+1)}(a)$. Therefore if $f^{(n+1)}(a)>0$, $f(a)$ is a minimum value of $f(x)$; and if $f^{(n+1)}(a)<0$, $f(a)$ is a maximum value of $f(x)$.

PROBLEMS

Find the values of the following definite integrals:

1. $\int_0^1 (x^3+3x+3)\,dx.$
2. $\int_3^4 \frac{dx}{x^2-4}.$
3. $\int_0^a (a^{\frac{1}{2}}-x^{\frac{1}{2}})^4\,dx.$
4. $\int_0^a \frac{x^3}{2a-x}\,dx.$
5. $\int_0^1 \left(1-\frac{x}{\sqrt{x^2+1}}\right)dx.$
6. $\int_0^1 xe^{-x^2}\,dx.$
7. $\int_0^{\frac{\pi}{6}} \tan x\,dx.$
8. $\int_0^{\frac{\pi}{3}} \frac{dx}{\cos x}.$
9. $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1+\sin\theta)^2\cos\theta\,d\theta.$
10. $\int_1^2 \frac{e^x+1}{e^x-1}\,dx.$
11. $\int_0^{\frac{1}{2}} \frac{e^x}{\cos^2 e^x}\,dx.$
12. $\int_{-\infty}^0 \frac{e^x\,dx}{\sqrt{1-e^{2x}}}.$
13. $\int_{-\infty}^{\infty} \frac{dx}{5+2x+x^2}.$
14. $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin^5\theta\,d\theta.$
15. $\int_0^{2\pi} \cos^3\frac{\theta}{4}\,d\theta.$
16. $\int_0^{\frac{\pi}{6}} \sin^2\phi\cos^3\phi\,d\phi.$
17. $\int_0^{\infty} \frac{dx}{(x^2+a^2)^2}.$
18. $\int_0^a \frac{dx}{(a^2+x^2)^{\frac{3}{2}}}.$
19. $\int_{-a}^a \sqrt{a^2-x^2}\,dx.$
20. $\int_a^x \sqrt{x^2-a^2}\,dx.$
21. $\int_0^{\pi} x\sin x\,dx.$
22. $\int_1^2 x^2\log x\,dx.$
23. $\int_0^{\frac{1}{2}} \sin^{-1}x\,dx.$
24. $\int_0^1 x^2\sin^{-1}x\,dx.$
25. $\int_0^1 x^2e^{-x}\,dx.$

26. Prove $\int_0^{\frac{\pi}{2}} \sin^6\theta\, d\theta = \frac{5}{6}\int_0^{\frac{\pi}{2}} \sin^4\theta\, d\theta$.

27. Prove $\int_0^{\frac{\pi}{2}} x^n \sin x\, dx = n\int_0^{\frac{\pi}{2}} x^{n-1} \cos x\, dx$, if $n > 1$.

28. Show that $\int_1^{\infty} \frac{dx}{x^k}$ has a definite value when, and only when, $k > 1$.

29. Show that $\int_a^b \frac{dx}{(b-x)^k}$ has a definite value when, and only when, $k < 1$.

30. If $f(-x) = -f(x)$, show graphically that $\int_{-a}^{a} f(x)\, dx = 0$.

31. If $f(-x) = f(x)$, show graphically that $\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$.

32. If $f(a-x) = f(x)$, show graphically that $\int_0^a f(x)\, dx = 2\int_0^{\frac{1}{2}a} f(x)\, dx$.

33. Show graphically that $\int_0^{2k\pi} f(\sin x)\, dx = k\int_0^{2\pi} f(\sin x)\, dx$.

34. If $a < b$, and $f_1(x)$, $f_2(x)$, $f_3(x)$ are three functions such that, for all values of x between a and b inclusive, $f_1(x) < f_2(x) < f_3(x)$, prove

$$\int_a^b f_1(x)\, dx < \int_a^b f_2(x)\, dx < \int_a^b f_3(x)\, dx.$$

35. Find the mean value of the lengths of the perpendiculars from a diameter of a semicircle to the circumference.

36. Find the mean value of the ordinates of the curve $y = \sin x$ between $x = 0$ and $x = \pi$.

37. A particle describes simple harmonic motion defined by the formula $s = a \sin kt$. Show that the mean kinetic energy during a complete vibration is half the maximum kinetic energy.

Obtain the following expansions. (The values of x for which the expansion is valid are given in each case.)

38. $\text{vers}\, x = \frac{x^2}{\underline{|2}} - \frac{x^4}{\underline{|4}} + \frac{x^6}{\underline{|6}} - \frac{x^8}{\underline{|8}} + \cdots$. $(-\infty < x < \infty)$

39. $\frac{1}{\sqrt{1+x^2}} = 1 - \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \cdots$. $(-1 < x < 1)$

40. $a^x = 1 + x\log a + \frac{(x\log a)^2}{\underline{|2}} + \frac{(x\log a)^3}{\underline{|3}} + \cdots$. $(-\infty < x < \infty)$

41. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots$. $\left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$

42. $\tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$. $(-1 < x < 1)$

43. $\log(x + \sqrt{1+x^2}) = x - \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7} + \cdots$. $-1 < x < 1$

44. $\sinh x = x + \dfrac{x^3}{\underline{|3}} + \dfrac{x^5}{\underline{|5}} + \dfrac{x^7}{\underline{|7}} + \cdots.$ $\quad (-\infty < x < \infty)$

45. $\cosh x = 1 + \dfrac{x^2}{\underline{|2}} + \dfrac{x^4}{\underline{|4}} + \dfrac{x^6}{\underline{|6}} + \cdots.$ $\quad (-\infty < x < \infty)$

46. Compute the value of $\tanh \frac{1}{2}$ to four decimal places.

47. Given $\log 2 = 0.693$, $\log 3 = 1.099$, what error would be made in assuming $\log 2\frac{1}{4} = \log 2 + \frac{1}{4}(1.099 - 0.693)$?

48. Assuming $\sin 60° = \frac{1}{2}\sqrt{3} = .8660$, $\cos 60° = \frac{1}{2}$, find $\sin 61°$ to four decimal places.

Verify the following expansions:

49. $\displaystyle\int_0^1 \frac{\log(1+x)}{x}\,dx = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots.$

50. $\displaystyle\int_0^1 \frac{\log x}{1-x}\,dx = -\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right).$

51. $\displaystyle\int_0^1 \frac{x^{a-1}}{1+x^b}\,dx = \frac{1}{a} - \frac{1}{a+b} + \frac{1}{a+2b} - \frac{1}{a+3b} + \cdots.$

52. $\displaystyle\int_0^x \cos x^2\,dx = x - \frac{x^5}{5\underline{|2}} + \frac{x^9}{9\underline{|4}} - \frac{x^{13}}{13\underline{|6}} + \cdots.$

CHAPTER IV

APPLICATIONS TO GEOMETRY

34. Element of a definite integral. In this and the subsequent chapter we shall give certain practical applications of the definite integral. Here we return in every case to the summation idea of § 21. The general method of handling one of the various problems proposed is to analyze it into the limit of the sum of an infinite number of infinitesimals of the form $f(x)\,dx$. The expression $f(x)\,dx$, as well as the concrete object it represents, is called the *element* of the sum.

35. Area of a plane curve in Cartesian coördinates. It has already been shown (§ 22) that the area bounded by the axis of x, the straight lines $x = a$ and $x = b$ $(a < b)$, and a portion of the curve $y = f(x)$ which lies above the axis of x is given by the definite integral

$$\int_a^b y\,dx. \tag{1}$$

It has also been noted that either of the bounding lines $x = a$ or $x = b$ may be replaced by a point in which the curve cuts OX. Here the element of integration $y\,dx$ represents the area of a rectangle with the base dx and the altitude y.

Similarly, the area bounded by the axis of y, the straight lines $y = c$ and $y = d$ $(c < d)$, and a portion of the curve $x = f(y)$ lying to the right of the axis of y is given by the integral

$$\int_c^d x\,dy, \tag{2}$$

where the element $x\,dy$ represents a rectangle with base x and altitude dy.

Areas bounded in other ways than these are frequently found by expressing the required area as the sum or the difference of areas of the above type.

Ex. 1. Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

It is evident from the symmetry of the curve (fig. 18) that one fourth of the required area is bounded by the axis of y, the axis of x, and the curve. Hence, if K is the total area of the ellipse,

$$K=4\int_0^a y\,dx=4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

$$=\frac{2b}{a}\left[x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}\right]_0^a=\pi ab.$$

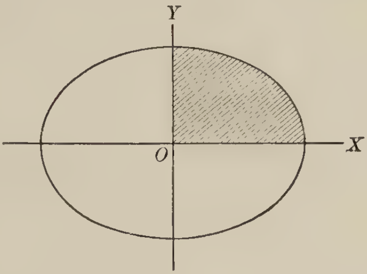

FIG. 18

Ex. 2. Find the area bounded by the axis of x, the parabola $y^2=4px$, and the straight line $y+2x-4p=0$ (fig. 19). The straight line and the parabola intersect at the point $C(p, 2p)$, and the straight line intersects OX at $B(2p, 0)$. The figure shows that the required area is the sum of two areas OCD and CBD. Hence, if K is the required area,

$$K=\int_0^p\sqrt{4px}\,dx+\int_p^{2p}(4p-2x)\,dx$$

$$=\left[\tfrac{4}{3}p^{\frac{1}{2}}x^{\frac{3}{2}}\right]_0^p+\left[4px-x^2\right]_p^{2p}=\tfrac{7}{3}p^2.$$

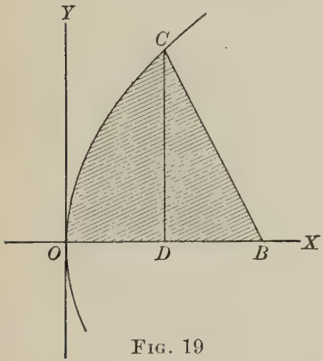

FIG. 19

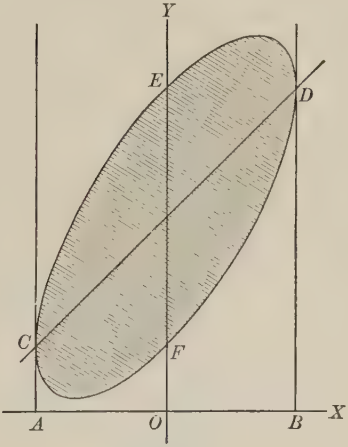

FIG. 20

Ex. 3. Find the area inclosed by the curve $(y-x-3)^2=4-x^2$.

Solving the equation for y, we have

$$y=x+3\pm\sqrt{4-x^2},$$

showing that the curve (fig. 20) lies between the straight lines $x=-2$ and $x=2$. It is clear from the figure that the area $ACEDB=\int_{-2}^{2}y_1\,dx$ and the area $ACFDB=\int_{-2}^{2}y_2\,dx$, where $y_1=x+3+\sqrt{4-x^2}$ and $y_2=x+3-\sqrt{4-x^2}$.

Therefore, if K is the required area $CEDFC$,

$$K = \int_{-2}^{2} y_1\,dx - \int_{-2}^{2} y_2\,dx = \int_{-2}^{2} (y_1 - y_2)\,dx = 2\int_{-2}^{2}\sqrt{4 - x^2}\,dx$$

$$= \left[x\sqrt{4 - x^2} + 4\sin^{-1}\frac{x}{2}\right]_{-2}^{2}$$

$$= 4\pi.$$

In the above examples we have replaced y in $\int_a^b y\,dx$ by its value $f(x)$ taken from the equation of the curve. More generally, if the equation of the curve is in the parametric form, we replace both x and y by their values in terms of the independent parameter. This is a substitution of a new variable, as explained in § 26, and the limits must be correspondingly changed.

Ex. 4. Let the equations of the ellipse be

$$x = a\cos\phi, \qquad y = b\sin\phi.$$

Then the area K of Ex. 1 may be computed as follows:

$$K = 4\int_0^a y\,dx = -4\int_{\frac{\pi}{2}}^{0} ab\sin^2\phi\,d\phi = 4ab\int_0^{\frac{\pi}{2}}\sin^2\phi\,d\phi = \pi ab.$$

Similarly, if the equation of the curve is, in polar coördinates, $r = f(\theta)$ and the area sought is one of the above forms, we may place

$$x = r\cos\theta = f(\theta)\cos\theta,$$
$$y = r\sin\theta = f(\theta)\sin\theta,$$

and obtain thus a parametric representation of the curve.

In case the axes of reference are oblique, the method of finding the area is easily modified, as follows:

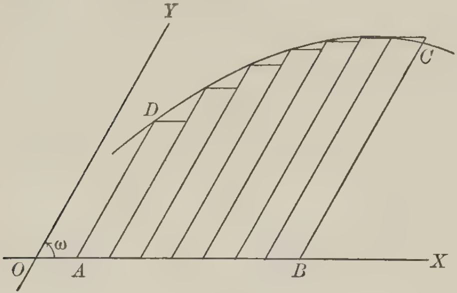

FIG. 21

Let the axes OX and OY (fig. 21) intersect at the angle ω, and let us find the area bounded by the curve $y = f(x)$, the axis of x, and the ordinates $x = a$ and $x = b$. The area is evidently the limit of the sum of the areas of the parallelograms whose sides are parallel to OX and OY. The area of one such parallelogram is $y\,\Delta x \sin\omega$. Hence the required area is

$$\sin\omega\int_a^b y\,dx. \tag{3}$$

36. Area of a plane curve in polar coördinates. Let O (fig. 22) be the pole, OM the initial line of a system of polar coördinates (r, θ), OA and OB two fixed radii vectors for which $\theta=\alpha$ and $\theta=\beta$ respectively, and AB any curve for which the equation is $r=f(\theta)$. Required the area AOB.

The required area may be divided into n smaller areas by dividing the angle $AOB=\beta-\alpha$ into n equal parts, each of which equals $\dfrac{\beta-\alpha}{n}=\Delta\theta$, and drawing the lines OP_1, OP_2, OP_3, $\cdots$, OP_{n-1}, where $AOP_1=P_1OP_2=P_2OP_3=\cdots=P_{n-1}OB=\Delta\theta$. (In the figure $n=8$.) The required area is the sum of the areas of these elementary areas for all values of n. The areas of these small figures, however, are no easier to find exactly than the given area, but we may find them approximately by describing from O as a center the circular arcs AR_1, P_1R_2, P_2R_3, $\cdots$, $P_{n-1}R_n$. Let

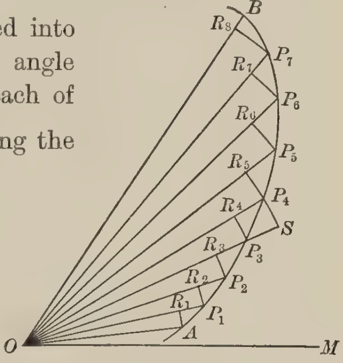

FIG. 22

$$OA=r_0,\quad OP_1=r_1,\quad OP_2=r_2,\quad \cdots,\quad OP_{n-1}=r_{n-1}.$$

Then, by geometry,

$$\begin{aligned}
&\text{the area of the sector } AOR_1=\tfrac{1}{2}r_0^2\,\Delta\theta,\\
&\text{the area of the sector } P_1OR_2=\tfrac{1}{2}r_1^2\,\Delta\theta,\\
&\quad.\qquad.\qquad.\qquad.\qquad.\qquad.\qquad.\qquad.\\
&\text{the area of the sector } P_{n-1}OR_n=\tfrac{1}{2}r_{n-1}^2\,\Delta\theta.
\end{aligned}$$

The sum of these areas, namely

$$\tfrac{1}{2}\sum_{i=0}^{i=n-1}r_i^2\Delta\theta,$$

is an approximation to the required area, and the limit of this sum as n is indefinitely increased is the required area.

To show this we need to show that the area of each of the circular sectors (e.g. P_3OR_4) differs from the corresponding elementary area (P_3OP_4) by an infinitesimal of higher order than either. For that purpose draw from P_4 an arc of a circle with center O intersecting OP_3 in S. Then

$$\text{area } P_3OR_4 < \text{area } P_3OP_4 < \text{area } SOP_4,$$

or

$$1 < \frac{\text{area } P_3OP_4}{\text{area } P_3OR_4} < \frac{\text{area } SOP_4}{\text{area } P_3OR_4}.$$

But area $P_3OR_4 = \frac{1}{2} r_3^2 \Delta\theta$ and area $SOP_4 = \frac{1}{2} r_4^2 \Delta\theta = \frac{1}{2}(r_3 + \Delta r)^2 \Delta\theta$.

Therefore $$\frac{\text{area } SOP_4}{\text{area } P_3OR_4} = \frac{(r_3 + \Delta r)^2}{r_3^2} = \left(1 + \frac{\Delta r}{r_3}\right)^2.$$

Now as n increases without limit, $\Delta\theta$ and consequently Δr approach zero as a limit. Hence $\lim\limits_{n=\infty} \dfrac{\text{area } SOP_4}{\text{area } P_3OR_4} = 1$, and therefore $\lim\limits_{n=\infty} \dfrac{\text{area } P_3OP_4}{\text{area } P_3OR_4} = 1$.

Hence the area P_3OR_4 differs from the area P_3OP_4 by an infinitesimal of higher order than either (§ 2), and therefore the limit of the sum of such areas as P_iOR_{i+1} equals the limit of the sum of such areas as P_iOP_{i+1}, as n is increased indefinitely (§ 3). But the latter limit is the area AOB, since

$$\sum_{i=0}^{i=n-1} P_iOP_{i+1} = \text{the area } AOB$$

for all values of n. Hence, finally,

$$\text{the area } AOB = \lim_{n=\infty} \tfrac{1}{2}\sum_{i=0}^{i=n-1} r_i^2 \Delta\theta = \frac{1}{2}\int_\alpha^\beta r^2 d\theta = \frac{1}{2}\int_\alpha^\beta [f(\theta)]^2 d\theta.$$

The student should compare this discussion with that of I, § 189.

The above result is unchanged if the point A coincides with O, but in that case OA must be tangent to the curve. So also B may coincide with O.

Ex. Find the area of one loop of the curve $r = a \sin 3\theta$ (I, § 177).

The required area K is given by the equation

$$K = \frac{a^2}{2}\int_0^{\frac{\pi}{3}} \sin^2 3\theta\, d\theta.$$

To integrate, place $3\theta = \phi$; then

$$K = \frac{a^2}{6}\int_0^{\pi} \sin^2\phi\, d\phi = \frac{\pi a^2}{12}.$$

37. Volume of a solid of revolution. *A solid of revolution is a solid generated by the revolution of a plane figure about an axis in its plane.* The simplest case is that in which the plane figure is bounded by the axis of revolution, two straight lines at right angles to the axis, and a curve which does not cut the axis. Such a solid is bounded by two parallel plane bases which are circles and by a *surface of revolution* generated by the revolving curve. Each point of this curve generates a circle whose center is in the axis of revolution and whose plane is parallel to the bases and perpendicular to the axis. Consequently, if planes are passed perpendicular to the axis, they will divide the solid into smaller solids with parallel circular bases. We shall proceed to find an approximate expression for the volume of one of these smaller solids.

Let KL (fig. 23) be the revolving curve, the equation of which is $x = f(y)$, OY the axis of revolution, CK the line $y = c$, and DL the line $y = d\,(c < d)$. Required the volume generated by the revolution of the figure $CKLD$. Divide the line CD into n equal parts, each of which equals $\frac{d-c}{n} = \Delta y$, by the points N_1, N_2, $N_3, \cdots, N_{n-1}$ where $ON_1 = y_1$, $ON_2 = y_2$, $ON_3 = y_3, \cdots, ON_{n-1} = y_{n-1}$.

Pass planes through the points N_1, N_2, N_3, $\cdots$, N_{n-1}, perpendicular to OY. They will intersect the surface of revolution in circles with the radii $x_1, x_2, x_3, \cdots, x_{n-1}$, where $x_1 = N_1P_1$, $x_2 = N_2P_2$, $x_3 = N_3P_3, \cdots, x_{n-1} = N_{n-1}P_{n-1}$. The areas of these sections, beginning with the base of the solid, are therefore πx_0^2, πx_1^2, $\pi x_2^2, \cdots$, πx_{n-1}^2, where $x_0 = CK$.

The solid is now cut into n slices of altitude Δy. We may consider the volume of each as approximately equal to that of a cylinder with its base coincident with the base of the slice and its altitude equal to that of the slice. The sum of the volumes of the n cylinders is

$$\pi x_0^2\Delta y + \pi x_1^2\Delta y + \pi x_2^2\Delta y + \cdots + \pi x_{n-1}^2\Delta y,$$

and the limit of this sum as n is indefinitely increased is, by definition, the volume of the solid of revolution.

This definition is seen to be reasonable and in accordance with the common conception of volume as follows. Whatever the definition of volume, it is evident that the volume of a solid is the sum of the volumes of its parts, and the volume of a solid inclosing another is greater than that of the solid inclosed.

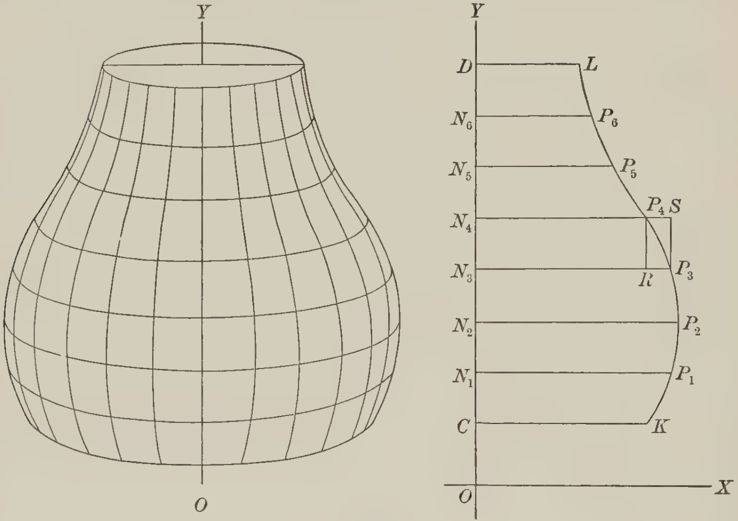

FIG. 23

Let then vol. $N_3N_4P_4P_3$ (for example) be the volume of the slice generated by the revolution of the plane figure $N_3N_4P_4P_3$ and draw the lines P_4R and P_3S parallel to OY. Then evidently

$$\text{vol. } N_3N_4SP_3 > \text{vol. } N_3N_4P_4P_3 > \text{vol. } N_3N_4P_4R$$

or
$$1 > \frac{\text{vol. } N_3N_4P_4P_3}{\text{vol. } N_3N_4SP_3} > \frac{\text{vol. } N_3N_4P_4R}{\text{vol. } N_3N_4SP_3}.$$

But
$$\text{vol. } N_3N_4SP_3 = \pi\overline{N_3P_3}^2 \cdot N_3N_4 = \pi x_3^2\Delta y$$

and
$$\text{vol. } N_3N_4P_4R = \pi\overline{N_4P_4}^2 \cdot N_3N_4 = \pi x_4^2\Delta y = \pi(x_3+\Delta x)^2\Delta y.$$

Hence $$\text{Lim}\,\frac{\text{vol. } N_3N_4P_4R}{\text{vol. } N_3N_4SP_3} = \text{Lim}\,\frac{(x_3+\Delta x)^2}{x_3^2} = 1,$$

and consequently $$\text{Lim}\,\frac{\text{vol. } N_3N_4P_4P_3}{\text{vol. } N_3N_4SP_3} = 1.$$

Therefore the volume of the slice differs from that of the cylinder by an infinitesimal of higher order than that of either (§ 2), and therefore the limit of the sum of the volumes of the slices is the same as that of the sum of the volumes of the cylinders (§ 3).

Hence, finally,

$$\text{vol. } CKLD = \text{Lim}_{n=\infty}\,\pi\sum_{i=0}^{i=n-1} x_i^2\Delta y = \pi\int_c^d x^2\,dy. \tag{1}$$

It is evident that the result is not invalidated if either L or K lies on OY.

Similarly, the volume generated by revolving about OX a figure bounded by OX, two straight lines $x = a$ and $x = b\,(a < b)$, and any curve not crossing OX is

$$\pi\int_a^b y^2\,dx. \tag{2}$$

To evaluate either of these integrals it is of course necessary to express x in terms of y, or y in terms of x, or both x and y in terms of a new variable, from the equation of the curve.

The volume of a solid generated by a plane figure of other shape than that just handled may often be found by taking the sum or the difference of two such volumes as the foregoing.

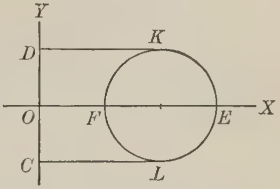

FIG. 24

Ex. Find the volume of the ring solid generated by revolving a circle of radius a about an axis in its plane b units from the center $(b > a)$.

Take the axis of revolution as OY (fig. 24) and a line through the center as OX. Then the equation of the circle is $(x - b)^2 + y^2 = a^2$.

The volume required is the difference between the volume generated by $CLEKD$ and that generated by $CLFKD$. But the volume generated by $CLEKD$ is $\pi\int_{-a}^{a} x_1^2\,dy$ where $x_1 = b + \sqrt{a^2 - y^2}$, and the volume generated by $CLFKD$ is $\pi\int_{-a}^{a} x_2^2\,dy$ where $x_2 = b - \sqrt{a^2 - y^2}$.

Therefore the required volume is

$$\pi\int_{-a}^{a}x_1^2\,dy-\pi\int_{-a}^{a}x_2^2\,dy=\pi\int_{-a}^{a}(x_1^2-x_2^2)\,dy$$

$$=4\,\pi b\int_{-a}^{a}\sqrt{a^2-y^2}\,dy=4\,\pi b\left[\frac{y}{2}\sqrt{a^2-y^2}+\frac{a^2}{2}\sin^{-1}\frac{y}{a}\right]_{-a}^{a}=2\,\pi^2a^2b.$$

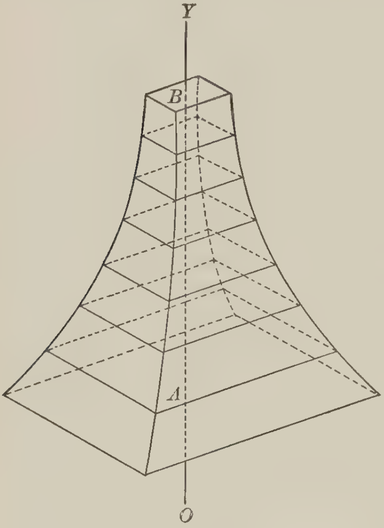

FIG. 25

38. Volume of a solid with parallel bases. Fig. 25 represents a solid with parallel bases. The straight line OY is drawn perpendicular to the bases, cutting the lower base at A, where $y=a$, and the upper base at B, where $y=b$. (The axis of x may be any line perpendicular to OY, but it is not shown in the figure and is not needed in the discussion.) Let the line AB be divided into n parts each equal to $\frac{b-a}{n}=\Delta y$, and let planes be passed through each point of division parallel to the bases of the solid. Let A_0 be the area of the lower base of the solid, A_1 the area of the first section parallel to the base, A_2 the area of the second section, and so on, A_{n-1} being the area of the section next below the upper base. Then $A_0\Delta y$ represents the volume of a cylinder with base equal to A_0 and altitude equal to Δy, $A_1\Delta y$ represents the volume of a cylinder standing on the next section as a base and extending to the section next above, and so forth. It is clear that

$$A_0\Delta y+A_1\Delta y+A_2\Delta y+\cdots+A_{n-1}\Delta y$$

is an approximation to the volume of the solid, and that the limit of this sum as n indefinitely increases is the volume of the solid. That this is rigorously true may be shown by a discussion similar to that of the previous article. That is, the required volume is

$$\int_a^b A\,dy.$$

To find the value of this integral it is necessary to express A in terms of y, or both A and y in terms of some other independent variable. This is a problem of geometry which must be solved for each solid. The solids of revolution are special cases. It is clear that the previous discussion is valid if the upper base reduces to a point, i.e. if the solid simply touches a plane parallel to its base. Similarly, both bases may reduce to points.

Ex. 1. Two ellipses with equal major axes are placed with their equal axes coinciding and their planes perpendicular. A variable ellipse moves so that its center is on the common axis of the given ellipses, the plane of the moving ellipse being perpendicular to those of the given ellipses. Required the volume of the solid generated.

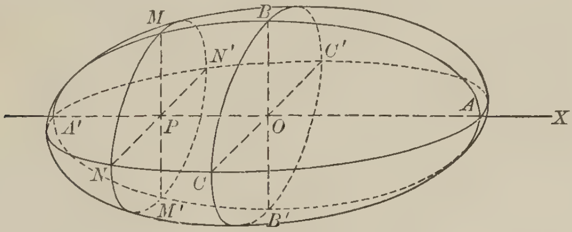

Fig. 26

Let the given ellipses be $ABA'B'$ (fig. 26) with semiaxes $OA = a$ and $OB = b$, and $ACA'C'$ with semiaxes $OA = a$ and $OC = c$, and let the common axis be OX. Let $NMN'M'$ be one position of the moving ellipse with the center P where $OP = x$. Then if A is the area of $NMN'M'$,

$$A = \pi \cdot PM \cdot PN. \qquad \text{(by § 35, Ex. 1)}$$

But from the ellipse $ABA'B'$ $\quad \dfrac{x^2}{a^2} + \dfrac{\overline{PM}^2}{b^2} = 1,$

and from the ellipse $ACA'C'$ $\quad \dfrac{x^2}{a^2} + \dfrac{\overline{PN}^2}{c^2} = 1.$

Therefore $\quad PM \cdot PN = \dfrac{bc}{a^2}(a^2 - x^2).$

Consequently the required volume is

$$\int_{-a}^{a} \frac{\pi bc}{a^2}(a^2 - x^2)\,dx = \frac{4}{3}\pi abc.$$

The solid is called an *ellipsoid* (§ 86, Ex. 5).

Ex. 2. The axes of two equal right circular cylinders intersect at right angles. Required the volume common to the cylinders.

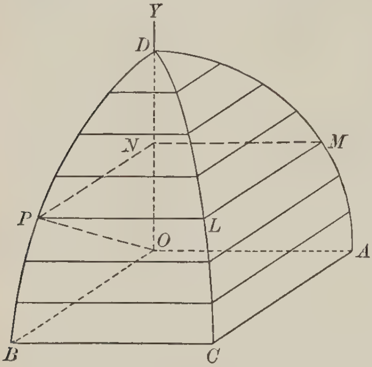

FIG. 27

Let OA and OB (fig. 27) be the axes of the cylinders, OY their common perpendicular at their point of intersection O, and a the radius of the base of each cylinder. Then the figure represents one eighth of the required volume V. A plane passed perpendicular to OY at a distance $ON = y$ from O intersects the solid in a square, of which one side is $NP = \sqrt{\overline{OP}^2 - \overline{ON}^2} = \sqrt{a^2 - y^2}$.

Therefore

$$\tfrac{1}{8} V = \int_0^a \overline{NP}^2 \, dy = \int_0^a (a^2 - y^2)\, dy = \tfrac{2}{3} a^3$$

and

$$V = \tfrac{16}{3} a^3.$$

39. The prismoidal formula. The formula $V = \int_a^b A\, dy$ leads to a simple and important result in those cases in which A can be expressed as a polynomial in y of degree not greater than 3. Let us place

$$A = a_0 y^3 + a_1 y^2 + a_2 y + a_3$$

and take O, for convenience, in the lower base of the solid. Then if h is the altitude of the solid,

$$\begin{aligned} V &= \int_0^h (a_0 y^3 + a_1 y^2 + a_2 y + a_3)\, dy \\ &= \tfrac{1}{4} a_0 h^4 + \tfrac{1}{3} a_1 h^3 + \tfrac{1}{2} a_2 h^2 + a_3 h. \end{aligned}$$

If now we place B for the area of the lower base, b for the area of the upper base, and M for the area of the section midway between the bases, we have

$$B = a_3,\ b = a_0 h^3 + a_1 h^2 + a_2 h + a_3,\ M = a_0\left(\frac{h}{2}\right)^3 + a_1\left(\frac{h}{2}\right)^2 + a_2\left(\frac{h}{2}\right) + a_3.$$

The formula for V then becomes

$$V = \frac{h}{6}(B + 4M + b).$$

This is known as the *prismoidal formula*.

To show its applicability to a given solid, we need only to show that the area of a cross section of the solid may be expressed as above. The most important and frequent cases are when A is a *quadratic* polynomial in y. In this way the student may show that the formula applies to frustra of pyramids, prisms, wedges, cones, cylinders, spheres, or solids of revolution in which the generating curve is a portion of a conic with one axis parallel to the axis of revolution, and also to the complete solids just named.

The formula takes its name, however, from its applicability to the solid called the *prismoid*, which we define as a solid having for its two ends dissimilar plane polygons with the same number of sides and the corresponding sides parallel, and for its lateral faces trapezoids.*

Furthermore, the formula is applicable to a more general solid, two of whose faces are plane polygons lying in parallel planes and whose lateral faces are triangles with their vertices in the vertices of these polygons.

Finally, if the number of sides of the polygons of the last defined solid is allowed to increase without limit, the solid goes over into a solid whose bases are plane curves in parallel planes and whose curved surface is generated by a straight line which touches each of the base curves. To such a solid the formula also applies (see § 91, Ex. 5).

The formula is extensively used by engineers in computing earthworks.

40. Length of a plane curve in rectangular coördinates. To find the length of any curve AB (fig. 28), assume $n-1$ points, $P_1, P_2, \cdots, P_{n-1}$, between A and B and connect each pair of consecutive points by a straight line. The length of AB is then defined as the limit of the sum of the lengths of the n chords $AP_1, P_1P_2, P_2P_3, \cdots, P_{n-1}B$ as n is increased without limit and the length of each chord approaches zero as a limit (I, § 104).

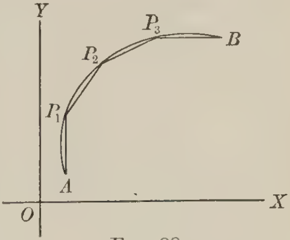

FIG. 28

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* The definition of the prismoid is variously given by different authors. We adopt that which connects the solid most closely with the prism.

Let the coördinates of P_i be (x_i, y_i) and those of P_{i+1} be $(x_i + \Delta x, y_i + \Delta y)$. Then the length of the chord P_iP_{i+1} is $\sqrt{(\Delta x)^2 + (\Delta y)^2}$, and the length of AB is the limit of the sum of the lengths of the n chords as n increases indefinitely. But $\sqrt{(\Delta x)^2 + (\Delta y)^2}$ is an infinitesimal which differs from the infinitesimal $\sqrt{dx^2 + dy^2}$ by an infinitesimal of higher order. For

$$\text{Lim}\,\frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\sqrt{dx^2 + dy^2}} = \text{Lim}\,\frac{\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \cdot \frac{\Delta x}{dx}.$$

Now if x is the independent variable, $\Delta x = dx$ (§ 4); if x is not the independent variable, $\text{Lim}\,\frac{\Delta x}{dx} = 1$ (§ 4). Also $\text{Lim}\,\frac{\Delta y}{\Delta x} = \frac{dy}{dx}$. Hence $\text{Lim}\,\frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\sqrt{dx^2 + dy^2}} = 1$. Therefore (§ 3), in finding the length of the curve, we may replace $\sqrt{(\Delta x)^2 + (\Delta y)^2}$ by $\sqrt{dx^2 + dy^2}$. Therefore if s is the length of AB, we have

$$s = \int_{(A)}^{(B)} \sqrt{dx^2 + dy^2}, \tag{1}$$

where (A) and (B) denote the values of the independent variable for the points A and B respectively.

If x is the independent variable, and the abscissas of A and B are a and b respectively, (1) becomes

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \tag{2}$$

If y is the independent variable, and the values of y at A and B are c and d respectively, (1) becomes

$$s = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy. \tag{3}$$

If x and y are expressed in terms of an independent parameter t, and the values of t for A and B respectively are t_0 and t_1, (1) becomes

$$s = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \tag{4}$$

Ex. 1. Find the length of the parabola $y^2 = 4px$ from the vertex to the point (h, k).

Formula (2) gives $$s = \int_0^h \sqrt{\frac{x+p}{x}}\,dx.$$

Formula (3) gives $$s = \frac{1}{2p}\int_0^k \sqrt{y^2 + 4p^2}\,dy.$$

Either integral leads to the result

$$s = \frac{k}{4p}\sqrt{k^2 + 4p^2} + p\log\frac{k + \sqrt{k^2 + 4p^2}}{2p}.$$

Ex. 2. Find the length of an ellipse.

If the equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and we measure the arc from the end of the minor axis, we have

$$\frac{1}{4}s = \int_0^a \sqrt{\frac{a^2 - e^2x^2}{a^2 - x^2}}\,dx,$$

where $e = \frac{\sqrt{a^2 - b^2}}{a}$, the eccentricity of the ellipse. Let us place $x = a\sin\phi$; then

$$\tfrac{1}{4}s = a\int_0^{\frac{\pi}{2}} \sqrt{1 - e^2\sin^2\phi}\,d\phi.$$

The indefinite integral cannot be found in terms of elementary functions. We therefore expand $\sqrt{1 - e^2\sin^2\phi}$ into a series by the binomial theorem (§ 31, Ex. 4); thus

$$\sqrt{1 - e^2\sin^2\phi} = 1 - \frac{1}{2}e^2\sin^2\phi - \frac{1}{2\cdot 4}e^4\sin^4\phi - \frac{1\cdot 3}{2\cdot 4\cdot 6}e^6\sin^6\phi - \cdots.$$

This series converges for all values of ϕ, since $e^2\sin^2\phi < 1$; then

$$\frac{1}{4}s = a\left\{\int_0^{\frac{\pi}{2}} d\phi - \frac{1}{2}e^2\int_0^{\frac{\pi}{2}}\sin^2\phi\,d\phi - \frac{1}{2\cdot 4}e^4\int_0^{\frac{\pi}{2}}\sin^4\phi\,d\phi - \frac{1\cdot 3}{2\cdot 4\cdot 6}e^6\int_0^{\frac{\pi}{2}}\sin^6\phi\,d\phi - \cdots\right\}$$

$$= \frac{a\pi}{2}\left\{1 - \left(\frac{1}{2}\right)^2 e^2 - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\frac{e^4}{3} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\frac{e^6}{5} - \cdots\right\}. \qquad \text{(by § 27, Ex. 2)}$$

The length of the ellipse may now be computed to any required degree of accuracy.

41. Length of a plane curve in polar coördinates. The formula

$$s = \int_{(A)}^{(B)} \sqrt{dx^2 + dy^2}$$

of § 40 may be transferred to polar coördinates by placing

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Then $$dx = \cos\theta\, dr - r\sin\theta\, d\theta,$$
$$dy = \sin\theta\, dr + r\cos\theta\, d\theta,$$
and $$dx^2 + dy^2 = dr^2 + r^2\, d\theta^2.$$

Therefore $$s = \int_{(A)}^{(B)} \sqrt{dr^2 + r^2\, d\theta^2}. \tag{1}$$

If θ is the independent variable, and the values of θ for A and B are α and β respectively, (1) becomes

$$s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \tag{2}$$

If r is the independent variable, and the values of r for A and B are a and b respectively, (1) becomes

$$s = \int_{a}^{b} \sqrt{1 + \left(r\frac{d\theta}{dr}\right)^2}\, dr. \tag{3}$$

42. The differential of arc. From

$$s = \int \sqrt{dx^2 + dy^2},$$

it follows (§ 9) that $$ds = \sqrt{dx^2 + dy^2}. \tag{1}$$

This relation between the differentials of x, y, and s is often represented by the triangle of fig. 29. This figure is convenient as a device for memorizing formula (1), but it should be borne in mind that RQ is not rigorously equal to dy (§ 5), nor is PQ rigorously equal to ds. In fact, $RQ = \Delta y$ and $PQ = \Delta s$, but since Δy and Δs differ from dy and ds respectively by infinitesimals of higher order, the use of RQ as dy and of PQ as ds is justified by the theorems of § 3. If now the triangle of fig. 29 is used as a plane right triangle, we have an easy method of recalling the formulas

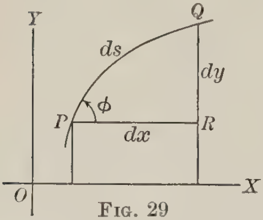

FIG. 29

$$ds^2 = dx^2 + dy^2,$$

$$\frac{dx}{ds} = \cos\phi, \qquad \frac{dy}{ds} = \sin\phi, \qquad \frac{dy}{dx} = \tan\phi.$$

Similarly, from $$s=\int\sqrt{dr^2+r^2d\theta^2}$$

we find $$ds=\sqrt{dr^2+r^2d\theta^2}, \qquad (2)$$

which is suggested by the triangle of fig. 30, where PQ is the arc of any curve and PR the arc of a circle with radius $OP=r$. This figure, if used as a straight-line figure, also gives the formulas

$$\tan\psi=\frac{r\,d\theta}{dr}, \qquad \cos\psi=\frac{dr}{ds},$$

$$\sin\psi=\frac{r\,d\theta}{ds}.$$

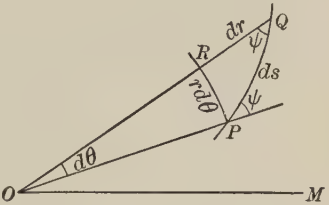

Fig. 30

43. Area of a surface of revolution. A surface of revolution is a surface generated by the revolution of a plane curve around an axis in its plane (§ 37). Let the curve AB (fig. 31) revolve about OY as an axis. To find the area of the surface generated, assume $n-1$ points, $P_1, P_2, P_3, \cdots, P_{n-1}$, between A and B, and connect each pair of consecutive points by a straight line. These lines are omitted in the figure since they are so nearly coincident with the arcs. The surface generated by AB is then defined as the limit of the sum of the areas of the surfaces generated by the n chords $AP_1, P_1P_2, P_2P_3, \cdots, P_{n-1}B$ as n increases without limit and the length of each chord approaches zero as a limit.

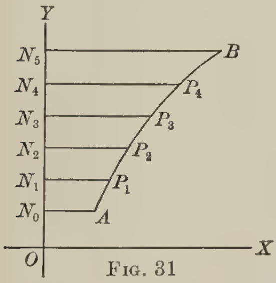

Fig. 31

Each chord generates the lateral surface of a frustum of a right circular cone, the area of which may be found by elementary geometry. Let the coördinates of P_i be (x_i, y_i) and those of P_{i+1} be $(x_i+\Delta x, y_i+\Delta y)$. Then the frustum of the cone generated by P_iP_{i+1} has for the radius of the upper base $N_{i+1}P_{i+1}$, for the radius of the lower base N_iP_i, and for its slant height P_iP_{i+1}. Its lateral area is therefore equal to

$$2\pi\frac{(N_iP_i+N_{i+1}P_{i+1})}{2}P_iP_{i+1}.$$

But $N_iP_i=x_i$, $N_{i+1}P_{i+1}=x_i+\Delta x$, $P_iP_{i+1}=\sqrt{(\Delta x)^2+(\Delta y)^2}$.

Therefore the lateral area of the frustum of the cone equals

$$2\pi\left(x_i + \frac{\Delta x}{2}\right)\sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

This is an infinitesimal which differs from

$$2\pi x_i\sqrt{dx^2 + dy^2} = 2\pi x_i ds$$

by an infinitesimal of higher order, and therefore the area generated by AB is the limit of the sum of an infinite number of these terms. Hence, if we represent the required area by S_y, we have

$$S_y = 2\pi\int_{(A)}^{(B)} x\, ds. \tag{1}$$

To evaluate the integral, we must either express ds in terms of x, or express x and ds in terms of y, or express both x and ds in terms of some other independent variable.

Similarly, the area generated by revolving AB about OX is

$$S_x = 2\pi\int_{(A)}^{(B)} y\, ds. \tag{2}$$

Ex. Find the area of the ring surface of the Ex., § 37.

The equation of the generating curve is

$$(x-b)^2 + y^2 = a^2,$$

and

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \frac{a}{\sqrt{a^2 - y^2}}\, dy.$$

The required area is the sum of the areas generated by the arcs LEK and LFK (fig. 24). Hence

$$S_y = 2\pi\int_{-a}^{a}(x_1 + x_2)\frac{a}{\sqrt{a^2-y^2}}\, dy = 4\pi ab\int_{-a}^{a}\frac{dy}{\sqrt{a^2-y^2}} = 4\pi^2 ab.$$

PROBLEMS

1. Find the area bounded by the axis of x and the parabola $x^2 - 16x + 4y = 0$.

2. Find the area included between a parabola and the tangents at the ends of the latus rectum. (The equation of the parabola referred to these tangents as axes is $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ (I, § 69).)

3. Find the total area bounded by the witch $y = \frac{8a^3}{x^2 + 4a^2}$ and its asymptote.

4. Find the area bounded by the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$, the axis of x, and the lines $x = \pm h$.

5. Find the total area of the curve $a^4y^2 + b^2x^4 = a^2b^2x^2$.

6. Find the area bounded by the curve $x^2y^2 + a^2b^2 = a^2y^2$ and its asymptotes.

7. Find the area bounded by the curve $y(x^2 + a^2) = a^2(a - x)$, the axis of x, and the axis of y.

8. Find the area of a segment of a circle of radius a cut off by a chord h units from the center.

9. Find the area contained in the loop of the curve $y^2 = x^2(a - x)$.

10. Find the area bounded by the curve $y^2(x^2 + a^2) = a^2x^2$ and its asymptotes.

11. Show that the area bounded by the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the axis of x, and the diameter through $P_1(x_1, y_1)$ on the curve is $\frac{ab}{2}\log\left(\frac{x_1}{a} + \frac{y_1}{b}\right)$.

12. Find the area bounded by the tractrix $y = \frac{a}{2}\log\frac{a + \sqrt{a^2 - x^2}}{a - \sqrt{a^2 - x^2}} - \sqrt{a^2 - x^2}$, the axis of x, and the axis of y.

13. Find the area between the axis of x and one arch of the cycloid $x = a(\phi - \sin\phi)$, $y = a(1 - \cos\phi)$.

14. Find the areas of each of the two portions into which the circle $x^2 + y^2 = 8$ is divided by the parabola $y^2 - 2x = 0$.

15. Find the area bounded by the parabola $x^2 - 4y = 0$ and the straight line $3x - 2y - 4 = 0$.

16. Find the area of the figure bounded by the parabolas $y^2 = 18x$ and $x^2 = \frac{16}{3}y$.

17. Find the area between the parabola $x^2 = 4ay$ and the witch $y = \frac{8a^3}{x^2 + 4a^2}$.

18. Find the total area of the lemniscate $r^2 = 2a^2\cos 2\theta$.

19. In the hyperbolic spiral $r\theta = a$ show that the area bounded by the spiral and two radii vectors is proportional to the difference of the length of the radii.

20. Find the area traversed by the radius vector of the spiral of Archimedes $r = a\theta$ in the first revolution.

21. Find the area described by the radius vector of $r = a\sec\theta$ from $\theta = \frac{\pi}{6}$ to $\theta = \frac{\pi}{3}$.

22. Find the total area of the cardioid $r = a(1 + \cos\theta)$.

23. Find the area of the limaçon $r = a\cos\theta + b$ when $b > a$.

24. Find the area bounded by the curves $r = a\cos 3\theta$ and $r = a$.

25. Find the area of a loop of the curve $r^2 = a^2\cos n\theta$.

26. Find the area of a loop of the curve $r = a\sin n\theta$.

27. Find the area of a loop of the curve $r\cos\theta = a\cos 2\theta$.

28. Find the area of the loop of the curve $r^2 = a^2 \cos 2\theta \cos\theta$ which is bisected by the initial line.

29. Find the total area of the curve $(x^2 + y^2)^2 = 4a^2x^2 + 4b^2y^2$. (Transform to polar coördinates.)

30. Find the area of the loop of the curve $(x^2 + y^2)^3 = 4a^2x^2y^2$. (Transform to polar coördinates.)

31. Find the volume generated by revolving about OY the surface bounded by the coördinate axes and the curve $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.

32. Find the volume of the solid generated by revolving about OY the plane surface bounded by OY and the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

33. Find the volume of the solid formed by revolving about OX the figure bounded by the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$, the axis of x, and the lines $x = \pm h$.

34. Find the volume of the solid formed by revolving about OY the plane figure bounded by the witch $y = \dfrac{8a^3}{x^2 + 4a^2}$ and the line $y = a$.

35. Find the volume of the solid formed by revolving about OX the plane figure bounded by the cissoid $y^2 = \dfrac{x^3}{2a - x}$, the line $x = a$, and the axis of x.

36. Find the volume of the solid generated by revolving about OY a segment of the circle $x^2 + y^2 = a^2$ cut off by the chord $x = h$.

37. Find the volume of the solid generated by revolving a semicircle of radius a around an axis parallel to the boundary diameter of the semicircle, (1) when the arc of the semicircle is concave toward the axis; (2) when the arc is convex toward the axis.

38. Find the volume of the solid formed by revolving an ellipse around its major axis.

39. Find the volume of the ring surface formed by revolving the ellipse $\dfrac{(x-d)^2}{a^2} + \dfrac{y^2}{b^2} = 1$ around $OY\,(d > a)$.

40. Find the volume of the solid generated by revolving about OX the surface bounded by the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and the line $x = a + h$.

41. Find the volume of the solid generated by revolving about OY the surface bounded by the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, the lines $y = \pm h$, and the axis of y.

42. Find the volume of the solid formed by revolving about the line $y = -a$ the figure bounded by the curve $y = \sin x$, the lines $x = 0$ and $x = \dfrac{\pi}{2}$, and the line $y = -a$.

43. Find the volume generated by revolving about the axis of x the figure bounded by the parabola $y^2 = 4px$ and the line $x = h$.

44. Find the volume of the solid formed by revolving about OY the figure bounded by the parabola $y^2 = 4px$, the axis of y, and the line $y = h$.

45. Find the volume of the solid formed by revolving about the line $x = -a$ the figure bounded by that line, the parabola $y^2 = 4px$, and the lines $y = \pm h$.

46. Find the volume of the solid formed by revolving about the line $x = a$ the figure bounded by the parabola $y^2 = 4px$ and the line $x = h\,(a > h)$.

47. A right circular cone has its vertex at the center of the sphere. Find the volume of the portion of the sphere intercepted by the cone.

48. A steel band is placed around a cylindrical boiler. A cross section of the band is a semiellipse, its axes being 6 and $\sqrt{6}$ in. respectively, the greater being parallel to the axis of the boiler. The diameter of the boiler is 48 in. What is the volume of the band?

49. Find the volume of the solid generated by revolving the cardioid $r = a(1 + \cos\theta)$ about the initial line.

50. Find the volume of the solid generated by revolving about OY the figure bounded by the axes of coördinates and the tractrix.

51. Prove by the method of § 38 that the volume of a cone with any base is equal to the product of one third the altitude by the area of the base.

52. Prove that the volume of a right conoid is equal to one half the product of its base and its altitude. (A *conoid* is a surface generated by a moving straight line which remains parallel to a fixed plane and intersects a fixed straight line. If the moving line is perpendicular to the fixed line, the conoid is a *right conoid*. The base is then the section made by a plane parallel to the fixed line, and the altitude is the distance of the fixed line from the plane of the base.)

53. On the double ordinate of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ an isosceles triangle is constructed with its altitude equal to the length of the ordinate. Find the volume generated as the triangle moves along the axis of the ellipse from vertex to vertex.

54. Find the volume cut from a right circular cylinder by a plane through the center of the base making an angle θ with the plane of the base.

55. Find the volume of the wedge-shaped solid cut from a right circular cylinder by two planes which pass through a diameter of the upper base and are tangent to the lower base.

56. Two circular cylinders with the same altitude have the upper base in common. Their other bases are tangent at the point where the perpendicular from the center of the upper base meets the plane of the lower bases. Find the volume common to the cylinders.

57. Two parabolas have a common vertex and a common axis but lie in perpendicular planes. An ellipse moves with its center on the common axis, its plane perpendicular to the axis, and its vertices on the parabolas. Find the volume generated when the ellipse has moved to a distance h from the common vertex of the parabolas.

58. A cylinder passes through great circles of a sphere which are at right angles to each other. Find the common volume.

59. A variable circle moves so that one point is always on OY, its center is always on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and its plane is always perpendicular to OY. Required the volume of the solid generated.

60. Two equal four-cusped hypocycloids are placed with their planes perpendicular and the straight line joining two opposite cusps of one in coincidence with a similar line in the other. A variable square moves with its plane perpendicular to this line and its vertices in the two curves. Find the volume of the solid generated.

61. Two equal ellipses are placed with their major axes in coincidence and their planes perpendicular. A straight line moves so as always to intersect the ellipses and to be parallel to a plane perpendicular to their common axis. Find the volume inclosed by the surface generated.

62. A variable rectangle moves so that one side has one end in OY and the other in the circle $x^2+y^2=a^2$. The ratio of the other side of the rectangle to the one mentioned is $\frac{p}{q}$, and the plane of the rectangle is perpendicular to OY. Required the volume of the solid generated.

63. The cap of a stone post is a solid of which every horizontal cross section is a square. The corners of all the squares lie in a spherical surface of radius 8 in. with its center 4 in. above the plane of the base. Find the volume of the cap.

64. Find the length of the semicubical parabola $y^2=x^3$ from the vertex to the point for which $x=h$.

65. Find the length of the curve $y=\log\frac{e^x+1}{e^x-1}$ from $x=1$ to $x=2$.

66. Find the total length of the four-cusped hypocycloid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

67. Find the length of the catenary from $x=0$ to $x=h$.

68. Find the length of the tractrix $y=\frac{a}{2}\log\frac{a+\sqrt{a^2-x^2}}{a-\sqrt{a^2-x^2}}-\sqrt{a^2-x^2}$ from $x=h$ to $x=a$.

69. Find the entire length of the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$.

70. Find the length of the curve $y^2=\frac{4}{27\,p}(x-2\,p)^3$ from $x=2\,p$ to $x=h$.

71. Find the length from cusp to cusp of the cycloid $x=a(\phi-\sin\phi)$, $y=a(1-\cos\phi)$.

72. Find the length of the epicycloid from cusp to cusp.

73. Find the length of the involute of a circle $x=a\cos\phi+a\phi\sin\phi$, $y=a\sin\phi-a\phi\cos\phi$ from $\phi=0$ to $\phi=\phi_1$.

74. From a spool of thread 2 in. in diameter three turns are unwound. If the thread is held constantly tight, what is the length of the path described by its end?

75. The cable of a suspension bridge hangs in the form of a parabola. The lowest point of the cable is 50 ft. above the water, and the points of suspension are 100 ft. above the water and 1000 ft. apart. Find the length of the cable.

76. Show that the length of the logarithmic spiral between any two points is proportional to the difference of the radii vectors of the points.

77. Find the length of the curve $r = a \cos^4 \frac{\theta}{4}$ from the point in which the curve intersects the initial line to the pole.

78. Find the complete length of the curve $r = a \sin^3 \frac{\theta}{3}$.

79. Find the length of the cardioid $r = a(1 + \cos\theta)$.

80. Find the area of a zone of a sphere bounded by the intersections of the sphere with two parallel planes at distances h_1 and h_2 from the center.

81. Find the area of the curved surface formed by revolving about OX the portion of the parabola $y^2 = 4px$ between $x = 0$ and $x = h$.

82. Find the area of the curved surface of the catenoid formed by revolving about OX the portion of the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ between $x = -h$ and $x = h$.

83. Find the area of the surface formed by revolving about OY the tractrix $y = \frac{a}{2} \log \frac{a + \sqrt{a^2 - x^2}}{a - \sqrt{a^2 - x^2}} - \sqrt{a^2 - x^2}$.

84. Find the area of the surface formed by revolving about OY the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

85. Find the area of the surface formed by revolving an arch of the cycloid $x = a(\phi - \sin\phi)$, $y = a(1 - \cos\phi)$ about OX.

86. Find the surface area of the oblate spheroid formed by revolving an ellipse about its minor axis.

87. Find the surface area of the prolate spheroid formed by revolving an ellipse around its major axis.

88. Find the area of the surface formed by revolving about the initial line the cardioid $r = a(1 + \cos\theta)$.

89. Find the area of the surface formed by revolving about the initial line the lemniscate $r^2 = 2a^2 \cos 2\theta$.

CHAPTER V

APPLICATIONS TO MECHANICS

44. Work. The application of the definite integral to determine the work done in moving a body in a straight line against a force in the same direction has been shown in § 21. Problems for the student will be found at the end of this chapter.

The case of a body moving in a curve, or acted on by forces not in the direction of the motion, is treated in Chap. XIV.

45. Attraction. Two particles of matter of masses m_1 and m_2 respectively, separated by a distance r, attract each other with a force equal to $k\frac{m_1 m_2}{r^2}$, where k is a constant which depends upon the units of force, distance, and mass. We shall assume that the units are so chosen that $k = 1$.

Let it now be required to find the attraction of a material body of mass m upon a particle of unit mass situated at a point A. Let the body be divided into n elements, the mass of each of which may be represented by Δm, and let P_i be a point at which the mass of one element may be considered as concentrated. Then the attraction of this element on the particle at A is $\frac{\Delta m}{r_i^2}$, where $r_i = P_iA$, and its component in the direction of OX is $\frac{\Delta m}{r_i^2}\cos\theta_i$, where θ_i is the angle between the directions P_iA and OX. The whole body, therefore, exerts upon the particle at A an attraction whose component in the direction OX is equal to

$$\operatorname*{Lim}_{n=\infty}\sum_{i=1}^{i=n}\frac{\cos\theta_i}{r_i^2}\Delta m.$$

If now $\cos\theta_i$, r_i, and Δm may be expressed in terms of a single independent variable, we have, for this component,

$$\operatorname*{Lim}_{n=\infty}\sum_{i=1}^{i=n}\frac{\cos\theta_i}{r_i^2}\Delta m = \int\frac{\cos\theta}{r^2}\,dm,$$

where the limits of integration are the values of the independent variable so chosen that the summation extends over the entire body. The manner in which this may be done in some cases is illustrated in the following example.

Ex. Find the attraction of a uniform wire of length l and mass m on a particle of unit mass situated in a straight line perpendicular to one end of the wire and at a distance a from it.

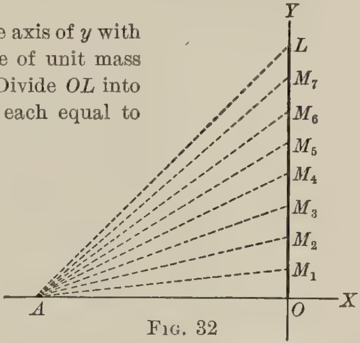

FIG. 32

Let the wire OL (fig. 32) be placed in the axis of y with one end at the origin, and let the particle of unit mass be at A on the axis of x where $AO = a$. Divide OL into n parts, OM_1, M_1M_2, M_2M_3, $\cdots$, $M_{n-1}L$, each equal to $\frac{l}{n} = \Delta y$. Then, if ρ is the mass per unit of length of the wire, the mass of each element is $\rho\Delta y = \Delta m$. We shall consider the mass of each element as concentrated at its first point, and shall in this way obtain an approximate expression for the attraction due to the element, this approximation being the better, the smaller is Δy. The attraction of the element M_iM_{i+1} on A is then approximately

$$\frac{\rho\Delta y}{\overline{AM_i}^2} = \frac{\rho\Delta y}{a^2 + y_i^2}, \text{ where } y_i = OM_i.$$

The component of this attraction in the direction OX is

$$\frac{\rho\Delta y}{a^2 + y_i^2}\cos OAM_i = \frac{\rho a\Delta y}{(a^2 + y_i^2)^{\frac{3}{2}}},$$

and the component in the direction OY is

$$\frac{\rho\Delta y}{a^2 + y_i^2}\sin OAM_i = \frac{\rho y_i\Delta y}{(a^2 + y_i^2)^{\frac{3}{2}}}.$$

Then, if X is the total component of the attraction parallel to OX, and Y the total component parallel to OY, we have

$$X = \operatorname*{Lim}_{n=\infty}\sum_{i=0}^{i=n-1}\frac{\rho a\Delta y}{(a^2 + y_i^2)^{\frac{3}{2}}} = \rho a\int_0^l \frac{dy}{(a^2 + y^2)^{\frac{3}{2}}},$$

$$Y = \operatorname*{Lim}_{n=\infty}\sum_{i=0}^{i=n-1}\frac{\rho y_i\Delta y}{(a^2 + y_i^2)^{\frac{3}{2}}} = \rho\int_0^l \frac{y\,dy}{(a^2 + y^2)^{\frac{3}{2}}}.$$

To verify this we may show that the true attraction of M_iM_{i+1} differs from the approximate attraction, which we have used, by an infinitesimal of higher order. Let I be the true x-component of the attraction of M_iM_{i+1}, I_1 the approximate attraction found by assuming that the whole mass of M_iM_{i+1} is at M_i, and I_2 the approximate attraction found by assuming that the whole mass of M_iM_{i+1} is at M_{i+1}. Then, evidently, $I_2 < I < I_1$; that is, $\frac{I_2}{I_1} < \frac{I}{I_1} < 1$. But $I_1 = \frac{\rho a \Delta y}{(a^2 + y_i^2)^{\frac{3}{2}}}$, $I_2 = \frac{\rho a \Delta y}{[a^2 + (y_i + \Delta y)^2]^{\frac{3}{2}}}$, and $\frac{I_2}{I_1} = \left[\frac{a^2 + y_i^2}{a^2 + (y_i + \Delta y)^2}\right]^{\frac{3}{2}}$. Therefore $\lim\limits_{n=\infty} \frac{I_2}{I_1} = 1$ and $\lim\limits_{n=\infty} \frac{I}{I_1} = 1$. Hence I_1 differs from I by an infinitesimal of higher order (§ 2), and may therefore be used in place of I in finding the limit of a sum (§ 3). A similar discussion may be given for the y-component.

To evaluate the integrals for X and Y, place $y = a \tan\theta$. Then, if $\alpha = \tan^{-1}\frac{l}{a} = OAL$,

$$X = \frac{\rho}{a}\int_0^{\alpha} \cos\theta \, d\theta = \frac{\rho}{a}\sin\alpha = \frac{m}{al}\sin\alpha,$$

$$Y = \frac{\rho}{a}\int_0^{\alpha} \sin\theta \, d\theta = \frac{\rho}{a}(1 - \cos\alpha) = \frac{m}{al}(1 - \cos\alpha),$$

since $l\rho = m$.

If R is the magnitude of the resultant attraction and β the angle which its line of action makes with OX,

$$R = \sqrt{X^2 + Y^2} = \frac{2m}{al}\sin\frac{1}{2}\alpha,$$

$$\beta = \tan^{-1}\frac{Y}{X} = \tan^{-1}\frac{1 - \cos\alpha}{\sin\alpha} = \frac{1}{2}\alpha.$$

46. Pressure. Consider a plane surface of area ΔA immersed in a liquid at a uniform depth of h units below the surface. The submerged surface supports a column of liquid of volume $h\Delta A$, the weight of which is $wh\Delta A$, where w (a constant for a given liquid) is the weight of a unit volume of the liquid. This weight is the total pressure on the immersed surface. The pressure per unit of area is then

$$\frac{wh\Delta A}{\Delta A} = wh,$$

which is independent of the size of ΔA. We may accordingly think of ΔA as infinitesimal and define wh as the pressure at a point h units below the surface. By the laws of hydrostatics this pressure is exerted equally in all directions. We may accordingly determine the pressure on plane surfaces which do not lie parallel to the surface of the liquid in the following manner.

Let OX (fig. 33) be in the surface of the liquid, and OH, for which the positive direction is downward, be the axis of h. Consider a surface $ABCD$ bounded by a portion of OH, two horizontal lines $AD(h=a)$ and $BC(h=b)$, and a curve with the equation $x=f(h)$. Divide AB into n segments, AM_1, M_1M_2, M_2M_3, $\cdots$, $M_{n-1}B$, each equal to $\frac{b-a}{n}=\Delta h$, and draw a line M_iP_i through each point M_i parallel to OX. Consider now the element of area $M_iP_iP_{i+1}M_{i+1}$, where $M_iP_i=x_i$ $=f(h_i)$. Its area is equal to that of a rectangle with base M_iM_{i+1} and altitude some line between M_iP_i and $M_{i+1}P_{i+1}$ (5, § 24), and is therefore equal to $(x_i+\epsilon_i)\Delta h$, where $\epsilon_i<\Delta x$. The pressure on the element would be $wh_i(x_i+\epsilon_i)\Delta h$, if all points of the element were at the depth $OM_i=h_i$, and would be $w(h_i+\Delta h)(x_i+\epsilon_i)\Delta h$, if all points were at the depth $OM_{i+1}=h_i+\Delta h$. Consequently, the pressure on $M_iP_iP_{i+1}M_{i+1}$ is $wh_ix_i\Delta h$ plus an infinitesimal of higher order. Therefore the total pressure P on the area $ABCD$ is

$$P=\operatorname*{Lim}_{n=\infty}\sum_{i=0}^{i=n-1}wh_ix_i\Delta h=\int_a^b whxdh=w\int_a^b hf(h)\,dh.$$

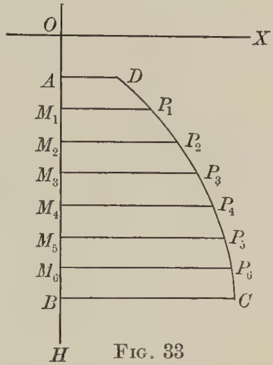

FIG. 33

The modification of this result necessary to adapt it to areas of slightly different shapes is easily made by the student and is exemplified in the following example.

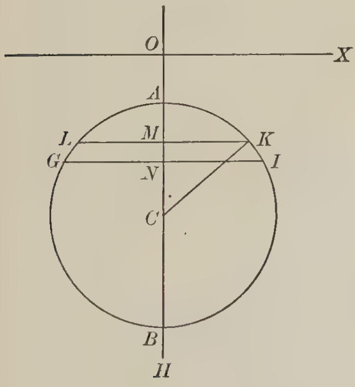

FIG. 34

Ex. Find the total pressure on a vertical circular area, a being the radius of the circle and b the depth of the center.

In fig. 34 let $OC=b$, $CA=CK=a$, $OM=h$, $MN=\Delta h$. Then $OA=b-a$, $OB=b+a$, $LK=2\,MK=2\sqrt{a^2-(h-b)^2}$, and the pressure on the strip $LKIG$ is $2\,wh\sqrt{a^2-(h-b)^2}\Delta h$, except for an infinitesimal of higher order. Therefore the total pressure on the circle is

$$P=2\,w\int_{b-a}^{b+a}\sqrt{a^2-(h-b)^2}\,hdh.$$

To integrate, place $h-b=a\sin\phi$; then

$$P=2\,w\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}a^2\cos^2\phi(b+a\sin\phi)d\phi=\pi a^2bw.$$

If the equation of the curve CD (fig. 33) is $x=f(y)$, referred to an origin at a depth c below the surface of the liquid, OX being parallel and OY perpendicular to the surface of the liquid, the pressure on $ABCD$ may be shown to be

$$w\int_a^b (c+y)f(y)\,dy,$$

a and b being the y coördinates of A and B respectively.

If this formula is used in the example, the center of the circle being taken as the origin of coördinates, we have

$$P=2w\int_{-a}^{a}(b+y)\sqrt{a^2-y^2}\,dy=\pi a^2bw.$$

47. Center of gravity. Consider n particles of masses m_1, m_2, m_3, $\cdots$, m_n, placed at the points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$, $\cdots$, $P_n(x_n, y_n)$ (fig. 35) respectively. The weights of these particles form a system of parallel forces equal to m_1g, m_2g, m_3g, $\cdots$, m_ng, where g is the acceleration due to gravity. The resultant of these forces is the total weight W of the n particles, where

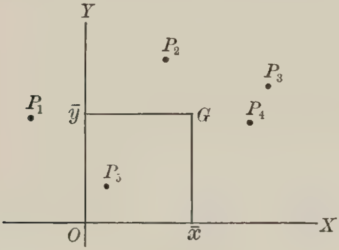

FIG. 35

$$W=m_1g+m_2g+m_3g+\cdots+m_ng$$
$$=g\sum_{i=1}^{i=n}m_i.$$

This resultant acts in a line which is determined by the condition that the moment of W about O is equal to the sum of the moments of the n weights.

Suppose first that gravity acts parallel to OY, and that the line of action of W cuts OX in a point the abscissa of which is $\bar{x}$. Then the moment of W about O is $g\bar{x}\sum m_i$ and the moment of one of the n weights is $g\,m_ix_i$.

Hence $$g\bar{x}\sum m_i=g\sum m_ix_i.$$

Similarly, if gravity acts parallel to OX, the line of action of the resultant cuts OY in a point the ordinate of which is $\bar{y}$, where

$$g\bar{y}\sum m_i=g\sum m_iy_i.$$

These two lines of action intersect in the point G, the coördinates of which are

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}, \qquad \bar{y} = \frac{\sum m_i y_i}{\sum m_i}. \tag{1}$$

Furthermore, if gravity acts in the XOY plane, but not parallel to either OX or OY, the line of action of its resultant always passes through G. This may be shown by resolving the weight of each particle into two components parallel to OX and OY respectively, finding the resultant of each set of components in the manner just shown, and then combining these two resultants.

If gravity acts in a direction not in the XOY plane, it may still be shown that its resultant acts through G, but the proof requires a knowledge of space geometry not yet given in this course.

The point G is called the center of gravity of the n particles.

If it is desired to find the center of gravity of a physical body, the problem may be formulated very roughly at first by saying that the body is made up of an infinite number of particles of matter each with an infinitesimal weight; hence the formulas for the coördinates of G must be extended to the case in which n is infinite. More precisely, the solution of the problem is as follows. The body in question is divided into n elementary portions such that the weight of each may be considered as concentrated at a point within it. If m is the total mass of the body, the mass of each element may be represented by Δm. Then if (x_i, y_i) are the coördinates of the point at which the mass of the ith element is concentrated, the center of gravity of the body is given by the equations

$$\bar{x} = \operatorname{Lim} \frac{\sum x_i \Delta m}{\sum \Delta m}, \qquad \bar{y} = \operatorname{Lim} \frac{\sum y_i \Delta m}{\sum \Delta m}.$$

In case x_i, y_i, and Δm can be expressed in forms of a single independent variable, these values become

$$\bar{x} = \frac{\int x \, dm}{\int dm}, \qquad \bar{y} = \frac{\int y \, dm}{\int dm}, \tag{2}$$

where the limits of integration are the values of the independent variable so chosen that the summation extends over the entire body.

It is to be noticed that it is not necessary, nor indeed always possible, to determine x_i, y_i exactly, since, by § 3,

$$\lim_{n=\infty}\sum_{i=1}^{i=n}(x_i+\epsilon_i)\,\Delta m=\lim_{n=\infty}\sum_{i=1}^{i=n}x_i\,\Delta m,$$

if ϵ_i approaches zero as Δm approaches zero.

The manner in which the operation thus sketched may be carried out in some cases is shown in the following articles. Discussion of the most general cases, however, must be postponed until after the subjects of double and triple integration are taken up.

48. Center of gravity of a plane curve. When we speak of the center of gravity of a plane curve we are to think of the curve as the axis of a rod, or wire, of uniform small cross section. Let LK (fig. 36) be such a curve, and let ρ be the amount of matter per unit of length of the wire which surrounds LK. That is, if s is the length of the wire and m is its mass, we have for a homogeneous wire

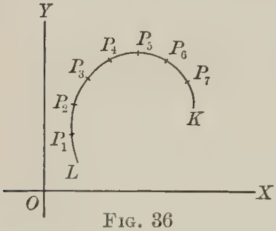

FIG. 36

$$\frac{\Delta m}{\Delta s}=\rho=\text{a constant};$$

and for a nonhomogeneous wire

$$\operatorname{Lim}\frac{\Delta m}{\Delta s}=\frac{dm}{ds}=\rho=\text{a function of } s.$$

The curve may now be divided into elements of arc, the length of each being Δs and its mass Δm, and formulas (2) of § 47 may be applied by placing

$$dm=\rho\,ds,$$

whence

$$\bar{x}=\frac{\int\rho x\,ds}{\int\rho\,ds},\qquad \bar{y}=\frac{\int\rho y\,ds}{\int\rho\,ds}. \tag{1}$$

The limits of integration are the values of the independent variable for L and K.

If ρ is constant, these formulas become

$$s\bar{x} = \int x\,ds, \qquad s\bar{y} = \int y\,ds, \tag{2}$$

where s is the length of LK.

Ex. 1. Find the center of gravity of a quarter circumference of the circle $x^2 + y^2 = a^2$, which lies in the first quadrant.

We have $$ds = \sqrt{dx^2 + dy^2} = \frac{a}{y}\,dx = -\frac{a}{x}\,dy.$$

Therefore $$\int x\,ds = -\int_a^0 a\,dy = a^2,$$

$$\int y\,ds = \int_0^a a\,dx = a^2,$$

and $$s = \frac{\pi a}{2}, \text{ a quarter circumference.}$$

Hence $$\bar{x} = \bar{y} = \frac{2a}{\pi}.$$

The problem may also be solved by using the parametric equations of the circle $x = a\cos\phi$, $y = a\sin\phi$.

Then $$\int x\,ds = a^2\int_0^{\frac{\pi}{2}}\cos\phi\,d\phi = a^2,$$

$$\int y\,ds = a^2\int_0^{\frac{\pi}{2}}\sin\phi\,d\phi = a^2.$$

Therefore $$\bar{x} = \bar{y} = \frac{2a}{\pi}, \text{ as before.}$$

Ex. 2. Find the center of gravity of a quarter circumference of a circle when the amount of matter in a unit of length is proportional to the length of the arc measured from one extremity.

We have here $\rho = ks$, where k is constant. Therefore, if we use the parametric equations of the circle,

$$\bar{x} = \frac{\int \rho x\,ds}{\int \rho\,ds} = \frac{\int sx\,ds}{\int s\,ds} = \frac{\int_0^{\frac{\pi}{2}} a^3\phi\cos\phi\,d\phi}{\int_0^{\frac{\pi}{2}} a^2\phi\,d\phi} = \frac{(4\pi - 8)a}{\pi^2},$$

$$\bar{y} = \frac{\int \rho y\,ds}{\int \rho\,ds} = \frac{\int sy\,ds}{\int s\,ds} = \frac{\int_0^{\frac{\pi}{2}} a^3\phi\sin\phi\,d\phi}{\int_0^{\frac{\pi}{2}} a^2\phi\,d\phi} = \frac{8a}{\pi^2}.$$

49. Center of gravity of a plane area. By the center of gravity of a plane area we mean the center of gravity of a thin sheet of matter having the plane area as its middle section. We shall first assume that the area is of the form of fig. 37, being bounded by the axis of x, the curve $y=f(x)$, and two ordinates. Divide the area into n elements by $n-1$ ordinates which divide AB into n parts each equal to Δx. We denote by ρ the mass per unit area. That is, if A is the area and m the mass, we have for a homogeneous sheet of matter

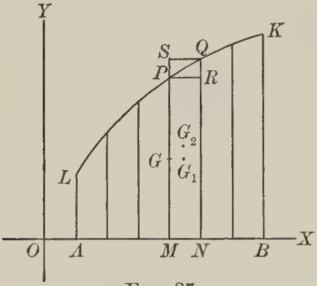

FIG. 37

$$\frac{\Delta m}{\Delta A}=\rho=\text{a constant};$$

and for a nonhomogeneous sheet of matter

$$\operatorname{Lim}\frac{\Delta m}{\Delta A}=\frac{dm}{dA}=\rho=\text{a variable}.$$

Hence in (2), § 47, we place

$$dm=\rho\, dA=\rho y\, dx,$$

since for the area in question $dA=y\,dx$ ((1), § 35).

Consider now any one of the elements $MNQP$, where $OM=x$, $ON=x+\Delta x$, $MP=y$, $NQ=y+\Delta y$, and draw the lines PR and QS parallel to OX. The mass of the rectangle $MNRP$ may be considered as concentrated at its middle point $G_1\left(x+\frac{\Delta x}{2}, \frac{y}{2}\right)$, and the mass of the rectangle $MNQS$ at its middle point $G_2\left(x+\frac{\Delta x}{2}, \frac{y}{2}+\frac{\Delta y}{2}\right)$. Accordingly the mass of $MNQP$ may be considered as concentrated at a point which lies above G_1, below G_2, and between the ordinates MP and NQ. The coördinates of any such point may be expressed as $\left(x+p\Delta x, \frac{y}{2}+q\frac{\Delta y}{2}\right)$, where $0 \leqq p \leqq 1$, $0 \leqq q \leqq 1$. But the coördinates of this point differ from those of $G\left(x, \frac{y}{2}\right)$ by an infinitesimal of the same order as Δx.

Therefore it is sufficient to consider the mass of $MNQP$ as concentrated at $G\left(x, \dfrac{y}{2}\right)\cdot$ Hence formulas (2), § 47, become

$$\bar{x} = \frac{\displaystyle\int \rho xy\, dx}{\displaystyle\int \rho y\, dx}, \qquad \bar{y} = \frac{\frac{1}{2}\displaystyle\int \rho y^2\, dx}{\displaystyle\int \rho y\, dx}. \tag{1}$$

If ρ is a constant, these formulas become

$$\bar{x} = \frac{\displaystyle\int xy\, dx}{\displaystyle\int y\, dx}, \qquad \bar{y} = \frac{\frac{1}{2}\displaystyle\int y^2\, dx}{\displaystyle\int y\, dx}, \tag{2}$$

which can be written

$$A\bar{x} = \int x\, dA, \qquad A\bar{y} = \tfrac{1}{2}\int y\, dA. \tag{3}$$

If it is required to find the center of gravity of a plane area of other shape than that just discussed, the preceding method may be modified in a manner illustrated by Ex. 2.

If the area has a line of symmetry, the center of gravity evidently lies upon it, and if this line is perpendicular to OX or OY, one of the coördinates of the center of gravity may be written down at once.

Ex. 1. Find the center of gravity of the area bounded by the parabola $y^2 = 4\,px$ (fig. 38), the axis of x, and the ordinate through a point (h, k) of the curve. Here

$$A\bar{x} = \int_0^h xy\, dx = 2\,p^{\frac{1}{2}}\int_0^h x^{\frac{3}{2}}\, dx = \tfrac{4}{5}\,p^{\frac{1}{2}}h^{\frac{5}{2}} = \tfrac{2}{5}\,h^2k,$$

$$A\bar{y} = \tfrac{1}{2}\int_0^h y^2\, dx = 2\,p\int_0^h x\, dx = ph^2 = \tfrac{1}{4}\,hk^2,$$

and $$A = \int_0^h y\, dx = 2\,p^{\frac{1}{2}}\int_0^h x^{\frac{1}{2}}\, dx = \tfrac{4}{3}\,p^{\frac{1}{2}}h^{\frac{3}{2}} = \tfrac{2}{3}\,hk.$$

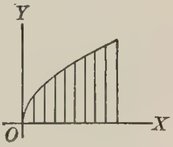

FIG. 38

Therefore $\bar{x} = \tfrac{3}{5}\,h, \qquad \bar{y} = \tfrac{3}{8}\,k.$

Ex. 2. Find the center of gravity of the segment of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (fig. 39) cut off by the chord through the positive ends of the axes of the curve. Divide the area into elements by lines parallel to OY. If we let y_2 be the ordinate of a point on the ellipse, and y_1 the ordinate of a point on the chord, we have as the element of area

$$dA = (y_2 - y_1)\,dx.$$

The mass of this element may be considered as concentrated at $\left(x,\ \frac{y_1+y_2}{2}\right)$.

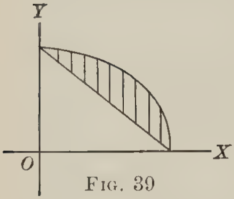

FIG. 39

Hence
$$\bar{x} = \frac{\int_0^a (y_2 - y_1)\,x\,dx}{\int_0^a (y_2 - y_1)\,dx},$$

$$\bar{y} = \frac{\frac{1}{2}\int_0^a (y_2^2 - y_1^2)\,dx}{\int_0^a (y_2 - y_1)\,dx}.$$

From the equation of the ellipse $y_2 = \frac{b}{a}\sqrt{a^2 - x^2}$, and from that of the chord $y_1 = \frac{b}{a}(a - x)$.

The denominator $\int_0^a (y_2 - y_1)\,dx$ is equal to the area of the quadrant of the ellipse minus that of a right triangle, i.e. is equal to $\frac{\pi ab}{4} - \frac{ab}{2}$.

Hence
$$\bar{x} = \frac{\frac{b}{a}\int_0^a x\left[\sqrt{a^2 - x^2} - (a - x)\right]dx}{ab\left(\frac{\pi}{4} - \frac{1}{2}\right)} = \frac{2a}{3(\pi - 2)},$$

$$\bar{y} = \frac{\frac{b^2}{2a^2}\int_0^a \left[(a^2 - x^2) - (a - x)^2\right]dx}{ab\left(\frac{\pi}{4} - \frac{1}{2}\right)} = \frac{2b}{3(\pi - 2)}.$$

50. Center of gravity of a solid or a surface of revolution of constant density. Consider a solid of revolution generated by revolving about OY the plane area bounded by OY, a curve $x = f(y)$, and two lines perpendicular to OY. The solid may be resolved into elements by passing planes perpendicular to OY at a distance dy apart (§ 37). The same planes divide the surface of revolution into elements (§ 43). If the density of the solid is uniform, it is evident from the symmetry of the figure that the mass either of an element of the solid or of the surface may be considered as lying in OY at the point where one of the planes which fixes the element cuts OY.

Consequently, the center of gravity lies in OY, so that

$$\bar{x} = 0.$$

Now if V is the volume of the solid and S the area of the surface, we have, by § 37 and § 43,

$$dV = \pi x^2 dy \quad \text{and} \quad dS = 2\,\pi x\, ds.$$

Consequently, to find the center of gravity of the solid we have to replace dm of § 47 (2) by $\rho\pi x^2 dy$ with the result

$$\bar{y} = \frac{\int \rho\pi x^2 y\, dy}{\int \rho\pi x^2 dy} = \frac{\int x^2 y\, dy}{\int x^2 dy}, \tag{1}$$

and to find the center of gravity of the surface of revolution we have to place in § 47 (2) $dm = 2\,\pi\rho x\, ds$ with the result

$$\bar{y} = \frac{\int 2\,\pi\rho x y\, ds}{\int 2\,\pi\rho x\, ds} = \frac{\int xy\, ds}{\int x\, ds}. \tag{2}$$

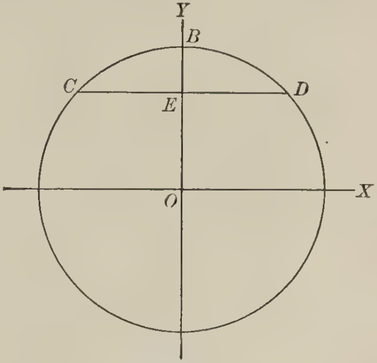

FIG. 40

Ex. 1. Find the center of gravity of a spherical segment of one base generated by revolving the area BDE (fig. 40) about OY.

Let $OB = a$ and $OE = c$. The equation of the circle is $x^2 + y^2 = a^2$,

and
$$\bar{y} = \frac{\int_c^a x^2 y\, dy}{\int_c^a x^2 dy} = \frac{\int_c^a (a^2 y - y^3)\, dy}{\int_c^a (a^2 - y^2)\, dy} = \frac{3}{4} \cdot \frac{(a+c)^2}{2\,a+c}.$$

Ex. 2. Find the center of gravity of the surface of the spherical segment of Ex. 1.

Using the notation and the figure of Ex. 1, we have $ds = \frac{a\, dy}{x}$, and therefore

$$\bar{y} = \frac{\int_c^a xy\, ds}{\int_c^a x\, ds} = \frac{\int_c^a y\, dy}{\int_c^a dy} = \frac{a+c}{2}.$$

The center of gravity lies half way between E and B.

51. Center of pressure. The pressures acting on the elements of a submerged plane area form a system of parallel forces perpendicular to the plane. The resultant of these forces is the total pressure P (§ 46) and the point at which it acts is the *center of pressure.* To find the center of pressure of an area of the type considered in § 46, we may proceed in a manner analogous to that used in finding the center of gravity of a plane area, only we have now to consider, instead of the weight of an element, the weight of the column of liquid which it sustains.

Let $(\bar{x}, \bar{h})$ be the coördinates of the center of pressure. Then the moment of the total pressure about OX is $P\bar{h}$, and the moment of the total pressure about OH is $P\bar{x}$. Also the pressure on an element $M_iM_{i+1}P_{i+1}P_i$ (fig. 33) is a force equal to $wh_ix_i\Delta h$ plus an infinitesimal of higher order (§ 46). By symmetry this force acts at the middle point of the element, the coördinates of which are $\frac{x_i}{2}$ and h_i, except for infinitesimals of higher order (§ 49). Hence the moment of this force about OX is $wh_i^2x_i\Delta h$ and the moment about OH is $\frac{1}{2}wh_ix_i^2\Delta h$, except for infinitesimals of higher order. Now the moment of the resultant must be equal to the sum of the moments of the component forces. Hence

$$P\bar{h} = \operatorname*{Lim}_{n=\infty}\sum_{i=0}^{i=n-1} wh_i^2x_i\Delta h = w\int_a^b h^2x\,dh,$$

$$P\bar{x} = \operatorname*{Lim}_{n=\infty}\sum_{i=0}^{i=n-1} \tfrac{1}{2}\,wh_ix_i^2\Delta h = \tfrac{1}{2}\,w\int_a^b hx^2dh.$$

Ex. Find the center of pressure of the circular area of the Ex., § 46.

By symmetry it is evident that $\bar{x} = 0$.

From the discussion just given,

$$P\bar{h} = 2\,w\int_{b-a}^{b+a}\sqrt{a^2 - (h-b)^2}\,h^2\,dh$$

$$= 2\,w\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^2\cos^2\phi\,(b + a\sin\phi)^2\,d\phi \quad (\text{where } h - b = a\sin\phi)$$

$$= 2\,a^2b^2w\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\phi\,d\phi + 4\,a^3bw\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\phi\sin\phi\,d\phi + \frac{a^4}{2}\,w\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2 2\,\phi\,d\phi$$

$$= \pi a^2b^2w + \tfrac{1}{4}\,\pi a^4w.$$

But $P = \pi a^2bw$ (Ex., § 46). Therefore $\bar{h} = b + \dfrac{a^2}{4\,b}$.

PROBLEMS

1. A positive charge m of electricity is fixed at O. The repulsion on a unit charge at a distance x from O is $\frac{m}{x^2}$. Find the work done in bringing a unit charge from infinity to a distance a from O.

2. A rod is stretched from its natural length a to the length $a+x$. Assuming that the force required in the stretching is proportional to $\frac{x}{a}$, find the work done.

3. A piston is free to slide in a cylinder of cross section S. The force acting on the piston is equal to pS, where p is the pressure of the gas in the cylinder. Find the work as the volume of the cylinder changes from v_1 to v_2, (1) assuming $pv = k$, (2) assuming $pv^\gamma = k$, γ and k being constants.

4. A spherical bag of radius a contains gas at a pressure equal to p_0 per unit of area. Assuming that the pressure per unit of area is inversely proportional to the volume occupied by the gas, show that the work required to compress the bag into a sphere of radius b is $4\pi a^3 p_0 \log \frac{a}{b}$.

5. The resistance offered by any conductor to the passage of a current of electricity is proportional to the distance traversed by the current in the conductor and inversely as the area of the cross section of the conductor. If a source of electricity is applied to the entire interior surface of a cylindrical shell, and the current flows radially outward, what resistance will be encountered? The length of the shell is h, the right circular section of the interior surface is of radius a and of the exterior surface is of radius b, and a unit cube of the substance of which the shell is made offers a resistance k.

6. A particle of unit mass is situated at a perpendicular distance c from the center of a straight homogeneous wire of mass M and length $2l$. Find the force of attraction exerted in a direction at right angles to the wire.

7. Find the attraction of a uniform straight wire of mass M upon a particle of unit mass situated in the line of direction of the wire at a distance c from one end.

8. Find the attraction of a uniform straight wire of mass M upon a particle of unit mass situated at a perpendicular distance c from the wire and so that lines drawn from the particle to the ends of the wire inclose an angle θ.

9. Find the attraction of a uniform circular wire of radius a and mass M upon a particle of unit mass situated at a distance c from the center of the ring in a straight line perpendicular to the plane of the ring.

10. Find the attraction of a uniform circular disk of radius a and mass M upon a particle of unit mass situated at a perpendicular distance c from the center of the disk. (Divide the disk into concentric rings and use the result of Ex. 9.)

11. Find the attraction of a uniform right circular cylinder with mass M, radius of its base a, and length l, upon a particle of unit mass situated in the axis of the cylinder produced, at a distance c from one end. (Divide the cylinder into parallel disks and use the result of Ex. 10.)

12. Find the attraction of a uniform wire of mass M bent into an arc of a circle with radius a and angle α, upon a particle of unit mass at the center of the circle.

13. Prove that the total pressure on a plane surface is equal to the pressure at the center of gravity multiplied by the area of the surface.

14. Find the total pressure on a vertical rectangle with base b and altitude a, submerged so that its upper edge is parallel to the surface of the liquid at a distance c from it.

15. Find the center of pressure of the rectangle in the previous example.

16. Find the total pressure on a triangle of base b and altitude a, submerged so that the base is horizontal, the altitude vertical, and the vertex in the surface of the liquid.

17. Show that the center of pressure of the triangle of the previous example lies in the median three fourths of the distance from the vertex to the base.

18. Find the total pressure on a triangle of base b and altitude a, submerged so that the base is in the surface of the liquid and the altitude vertical.

19. Show that the center of pressure of the triangle of the previous example lies in the median half way from the vertex to the base.

20. Find the total pressure on an isosceles triangle with base $2b$ and altitude a, submerged so that the base is horizontal, the altitude vertical, and the vertex, which is above the base, at a distance c from the surface of the liquid.

21. A parabolic segment with base $2b$ and altitude a is submerged so that its base is horizontal, its axis vertical, and its vertex in the surface of the liquid. Find the total pressure.

22. Find the center of pressure of the parabolic segment of the previous example.

23. A parabolic segment with base $2b$ and altitude a is submerged so that its base is in the surface of the liquid and its altitude is vertical. Find the total pressure.

24. Find the center of pressure of the parabolic segment of the previous example.

25. Find the total pressure on a semiellipse submerged with one axis in the surface of the liquid and the other vertical.

26. Find the center of pressure of the ellipse of the previous example.

27. An isosceles triangle with its base horizontal and vertex downward is immersed in water*. Find the pressure on the triangle if the length of the base is 8 ft., the altitude 3 ft., and the depth of the vertex below the surface 5 ft.

28. The centerboard of a yacht is in the form of a trapezoid in which the two parallel sides are 1 and 2 ft. respectively in length, and the side perpendicular to these two is 3 ft. in length. Assuming that the last-named side is parallel to the surface of the water at a depth of 2 ft., and that the parallel sides are vertical, find the pressure on the board.

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* The weight of a cubic foot of water may be taken as $62\frac{1}{2}$ lb.

29. Find the moment of the force which tends to turn the centerboard of the previous example about the line of intersection of the plane of the board with the surface of the water.

30. Find the pressure on the centerboard of Ex. 28 if the plane of the board is turned through an angle of 10° about its line of intersection with the surface of the water.

31. A dam is in the form of a regular trapezoid with its two horizontal sides 300 and 100 ft. respectively, the longer side being at the top and the height 20 ft. Assuming that the water is level with the top of the dam, find the total pressure.

32. Find the moment of the force which tends to overturn the dam of Ex. 31 by turning it on its base line.

33. A circular water main has a diameter of 6 ft. One end is closed by a bulkhead and the other is connected with a reservoir in which the surface of the water is 100 ft. above the center of the bulkhead. Find the total pressure on the bulkhead.

34. A pond of 10 ft. depth is crossed by a roadway with vertical sides. A culvert, whose cross section is in the form of a parabolic segment with horizontal base on a level with the bottom of the pond, runs under the road. Assuming that the base of the segment of the parabolic segment is 6 ft. and its altitude 4 ft., find the total pressure on the bulkhead which temporarily closes the culvert.

35. Find the center of gravity of the semicircumference of the circle $x^2 + y^2 = a^2$ which is above the axis of x.

36. Find the center of gravity of the arc of the four-cusped hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ which is in the first quadrant.

37. Find the center of gravity of the arc of the four-cusped hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ which is above the axis of x.

38. Find the center of gravity of the arc of the curve $9ay^2 - x(x - 3a)^2 = 0$ between the ordinates $x = 0$ and $x = 3a$.

39. Find the center of gravity of the area bounded by a parabola and a chord perpendicular to the axis.

40. Find the center of gravity of the area bounded by the semicubical parabola $ay^2 = x^3$ and any double ordinate.

41. Find the center of gravity of the area of a quadrant of an ellipse.

42. Find the center of gravity of the area between the axes of coördinates and the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.

43. Find the center of gravity of the area contained in the upper half of the loop of the curve $ay^2 = ax^2 - x^3$.

44. Show that the center of gravity of a sector of a circle lies on the line bisecting the angle of the sector at a distance $\frac{2}{3}a\frac{\sin\frac{\alpha}{2}}{\frac{\alpha}{2}}$ from the vertex, where α is the angle and a the radius of the sector.

45. Find the center of gravity of the area bounded by the curve $y = \sin x$ and the axis of x between $x = 0$ and $x = \pi$.

46. If the area to the right of the axis of y between the curve $y = \frac{h}{\sqrt{\pi}} e^{-h^2x^2}$ and the axis of x is $\frac{1}{2}$, what is the abscissa of the center of gravity of this area?

47. Find the center of gravity of a triangle.

48. Find the center of gravity of the area between the parabola $y^2 = 4px$ and the straight line $y = mx$.

49. Find the center of gravity of the plane area bounded by the two parabolas $y^2 = 4px$, and $x^2 = 4py$.

50. Find the center of gravity of the area bounded by the two parabolas $x^2 - 4p(y - b) = 0$, $x^2 - 4py = 0$, the axis of y, and the line $x = a$.

51. Find the center of gravity of the plane area common to the parabola $x^2 - 4py = 0$ and the circle $x^2 + y^2 - 32p^2 = 0$.

52. Find the center of gravity of the surface bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the circle $x^2 + y^2 = a^2$, and the axis of y.

53. Find the center of gravity of half a spherical solid of constant density.

54. Find the center of gravity of the portion of a spherical surface bounded by two parallel planes at a distance h_1 and h_2 respectively from the center.

55. Find the center of gravity of the solid formed by revolving about OX the surface bounded by the parabola $y^2 = 4px$, the axis of x, and the line $x = a$.

56. Find the center of gravity of the solid formed by revolving about OY the plane figure bounded by the parabola $y^2 = 4px$, the axis of y, and the line $y = k$.

57. Find the center of gravity of the solid generated by revolving about the line $x = a$ the surface bounded by that line, the axis of x, and the parabola $y^2 = 4px$.

58. Find the center of gravity of the solid formed by revolving about OY the surface bounded by the parabola $x^2 = 4py$ and any straight line through the vertex.

59. Find the center of gravity of the solid formed by revolving about OY the surface bounded by the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and the lines $y = 0$ and $y = b$.

60. Find the center of gravity of a hemispherical surface.

61. Find the center of gravity of the surface of a right circular cone.

62. Find the center of gravity of the surface of a hemisphere when the density of each point in the surface varies as its perpendicular distance from the circular base of the hemisphere.

CHAPTER VI

INTEGRATION OF RATIONAL FRACTIONS

52. Introduction. The sum

$$\frac{A_1}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+\frac{A_3x+B_3}{a_3x^2+b_3x+c_3},$$

each term of which is a rational fraction in its lowest terms with the degree of the numerator less than that of the denominator, is known by elementary algebra to be a fraction of the form $\frac{f(x)}{F(x)}$, where

$$F(x)=(a_1x+b_1)(a_2x+b_2)(a_3x^2+b_3x+c_3)$$

and $$f(x)=A_1(a_2x+b_2)(a_3x^2+b_3x+c_3)+A_2(a_1x+b_1)(a_3x^2+b_3x+c_3) + (A_3x+B_3)(a_1x+b_1)(a_2x+b_2).$$

Again, consider the sum

$$\frac{A_1}{(a_1x+b_1)^2}+\frac{A_1'}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+\frac{A_3x+B_3}{(a_3x^2+b_3x+c_3)^2}+\frac{A_3'x+B_3'}{a_3x^2+b_3x+c_3}.$$

Here the linear polynomial a_1x+b_1 appears both in the first and the second powers as denominators of fractions which have the same form of numerator, a constant; also the quadratic polynomial $a_3x^2+b_3x+c_3$ appears both in the first and the second powers as denominators of fractions which have the same form of numerator, a linear polynomial. If this sum is denoted by $\frac{f(x)}{F(x)}$, then

$$F(x)=(a_1x+b_1)^2(a_2x+b_2)(a_3x^2+b_3x+c_3)^2,$$

and $f(x)$, when determined, will be of lower degree than $F(x)$.

In both examples, $f(x)$ and $F(x)$ have no common factor and $f(x)$ is of lower degree than $F(x)$.

We proceed in the following articles to consider, conversely, the possibility of separating any rational fraction $\frac{f(x)}{F(x)}$ in which $f(x)$ is of lower degree than $F(x)$ into a sum of fractions of the types we have just added.

53. Separation into partial fractions. Consider now any rational fraction $\frac{f(x)}{F(x)}$, where $f(x)$ and $F(x)$ are two polynomials having no common factor. If the degree of $f(x)$ is not less than that of $F(x)$, we can separate the fraction, by actual division, into an integral expression and a fraction in which the degree of the numerator is less than that of the denominator.

For example, by actual division,

$$\frac{2x^5 + x^4 + x^3 + x^2 - 18x - 6}{x^4 - 9} = 2x + 1 + \frac{x^3 + x^2 + 3}{x^4 - 9}.$$

Accordingly, we shall consider only the case in which the degree of $f(x)$ is less than that of $F(x)$.

Now $F(x)$ is always equivalent to the product of linear factors (I, § 42), which are not necessarily real; and if the coefficients of $F(x)$ are real, it is equivalent to the product of real linear and quadratic factors (I, § 45). We shall limit ourselves in this chapter to polynomials with real coefficients and shall assume that the real linear and quadratic factors of $F(x)$ can be found. We shall make two cases:

Case I, where no factor is repeated.

Case II, where some of the factors are repeated.

Case I. As an example of this case let

$$F(x) = (a_1x + b_1)(a_2x + b_2)(a_3x^2 + b_3x + c_3).$$

May we then assume, as suggested by the work of the previous article, that

$$\frac{f(x)}{F(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \frac{A_3x + B_3}{a_3x^2 + b_3x + c_3}, \tag{1}$$

where A_1, A_2, A_3, and B_3 are constants?

It is evident that the sum of the fractions in the right-hand member of (1) is a fraction the denominator of which is $F(x)$, and the numerator of which is a polynomial, which, like $f(x)$, is of lower degree than $F(x)$.

It will be proved in §§ 55, 56 that A_1, A_2, A_3, and B_3 exist. Assuming this, we may multiply both sides of (1) by $F(x)$, with the following result:

$$f(x) = A_1(a_2x + b_2)(a_3x^2 + b_3x + c_3) + A_2(a_1x + b_1)(a_3x^2 + b_3x + c_3) + (A_3x + B_3)(a_1x + b_1)(a_2x + b_2). \tag{2}$$

As (2) is to hold for all values of x, the coefficients of like powers of x on the two sides of the equation must be equal.* The right-hand member is of degree three, and by hypothesis the left-hand member of degree no higher than three. Hence, placing the coefficients of x^3, x^2, x, and the constant term on the two sides of the equation respectively equal, we have four equations from which to find the four unknown constants A_1, A_2, A_3, B_3.

Solving these equations and substituting the values of A_1, A_2, A_3, and B_3 in (1), we have the original fraction expressed as the sum of three fractions, the denominators of which are the factors of the denominator of the original fraction. The fraction is now said to be separated into *partial fractions.*

It is evident that the number of the factors of $F(x)$ in no way affects the reasoning or the conclusion, and there will always be the same number of equations as the number of the unknown constants to be determined.

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* If the two members of the equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = b_0x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n \tag{1}$$

are identical, so that (1) is true for all values of x, the coefficients of like powers of x on the two sides of (1) are equal, i.e. $a_0 = b_0$, $a_1 = b_1$, $\cdots$, $a_n = b_n$.

Writing (1) in the equivalent form

$$(a_0 - b_0)x^n + (a_1 - b_1)x^{n-1} + \cdots + (a_{n-1} - b_{n-1})x + (a_n - b_n) = 0, \tag{2}$$

we have an algebraic equation of degree not greater than n, unless $a_0 = b_0$, $a_1 = b_1$, $\cdots$, $a_n = b_n$.

Then (2) is true only for a certain number of values of x, since the number of roots of an algebraic equation is the same as the degree of the equation. But this is contrary to the hypothesis that (1), and therefore (2), is true for all values of x. Hence $a_0 = b_0$, $a_1 = b_1$, $\cdots$, $a_n = b_n$, as was stated.

Ex. 1. Separate into partial fractions $\frac{x^2+11x+14}{(x+3)(x^2-4)}$.

Since the degree of the numerator is less than the degree of the denominator, we assume

$$\frac{x^2+11x+14}{(x+3)(x^2-4)}=\frac{A}{x-2}+\frac{B}{x+2}+\frac{C}{x+3}, \tag{1}$$

where A, B, and C are constants.

Clearing (1) of fractions by multiplying by $(x+3)(x^2-4)$, we have

$$x^2+11x+14=A(x+2)(x+3)+B(x-2)(x+3)+C(x-2)(x+2), \tag{2}$$

or $$x^2+11x+14=(A+B+C)x^2+(5A+B)x+(6A-6B-4C). \tag{3}$$

Since (3) is to hold for all values of x, the coefficients of like powers of x on the two sides of the equation must be equal.

Therefore
$$\begin{aligned} A+B+C&=1,\\ 5A+B&=11,\\ 6A-6B-4C&=14, \end{aligned}$$

whence we find $A=2$, $B=1$, and $C=-2$.

Substituting these values in (1), we have

$$\frac{x^2+11x+14}{(x+3)(x^2-4)}=\frac{2}{x-2}+\frac{1}{x+2}-\frac{2}{x+3}.$$

If the factors of the denominator are all linear and different, as in this example, the following special method is of decided advantage. In (2) let x have in succession such a value as to make one of the factors of the denominator of the original fraction zero, i.e. $x=2$, $x=-2$, $x=-3$.

When $x=2$, (2) becomes $40=20A$, whence $A=2$; when $x=-2$, (2) becomes $-4=-4B$, whence $B=1$; and when $x=-3$, (2) becomes $-10=5C$, whence $C=-2$.

The method just used may seem to be invalid in that (2) apparently holds for all values of x except 2, -2, and -3, since these values make the multiplier $(x+3)(x^2-4)$, by which (2) was derived from (1), zero. This objection is met, however, by considering that the two polynomials in (2) are identical and therefore equal for all values of x, including the values 2, -2 and -3.

Ex. 2. Separate into partial fractions $\frac{x^3+4x^2+x}{x^3-1}$.

Since the degree of the numerator is not less than that of the denominator, we divide until the degree of the remainder is less than the degree of the divisor, and thus find

$$\frac{x^3+4x^2+x}{x^3-1}=1+\frac{4x^2+x+1}{x^3-1}. \tag{1}$$

The real factors of x^3-1 are $x-1$ and x^2+x+1. Hence we assume

$$\frac{4x^2+x+1}{x^3-1}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}. \tag{2}$$

Clearing of fractions, we have

$$\begin{aligned}4x^2 + x + 1 &= A(x^2 + x + 1) + (Bx + C)(x - 1)\\ &= (A + B)x^2 + (A - B + C)x + (A - C). \qquad (3)\end{aligned}$$

Equating coefficients of like powers of x in (3), we obtain the equations

$$\begin{aligned}A + B &= 4,\\ A - B + C &= 1,\\ A - C &= 1,\end{aligned}$$

whence $$A = 2, \qquad B = 2, \qquad C = 1.$$

Hence $$\frac{4x^2 + x + 1}{x^3 - 1} = \frac{2}{x - 1} + \frac{2x + 1}{x^2 + x + 1},$$

and $$\frac{x^3 + 4x^2 + x}{x^3 - 1} = 1 + \frac{2}{x - 1} + \frac{2x + 1}{x^2 + x + 1}.$$

The values of A, B, and C may also be found by assuming arbitrary values of x. Thus when $x = 1$, (3) becomes $6 = 3A$; when $x = 0$, (3) becomes $1 = A - C$; and when $x = 2$, (3) becomes $19 = 7A + 2B + C$; whence $A = 2$, $C = 1$, $B = 2$.

54. Case II. We will now consider the case in which some of the factors of the denominator $F(x)$ are repeated. For example, let

$$F(x) = (a_1x + b_1)^2 (a_2x + b_2)(a_3x^2 + b_3x + c_3)^2.$$

We assume

$$\frac{f(x)}{F(x)} = \frac{A_1}{(a_1x + b_1)^2} + \frac{A_1'}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \frac{A_3x + B_3}{(a_3x^2 + b_3x + c_3)^2} + \frac{A_3'x + B_3'}{a_3x^2 + b_3x + c_3}, \qquad (1)$$

as suggested by the work of § 52.

Multiplying (1) by $F(x)$, we have an equation of the 6th degree in x, since the degree of $f(x)$ is, by hypothesis, less than that of $F(x)$. Equating the coefficients of x^6, x^5, x^4, x^3, x^2, x, and the constant term on the two sides of the equation, we have seven equations from which to determine the seven unknown constants, A_1, A_1', $\ldots$, B_3'.

It is evident that, granted the existence of these constants, the above method for determining them is perfectly general.

Ex. 1. Separate into partial fractions $\frac{x^4-6x^2+16}{(x+2)(x^2-4)}$.

Since the degree of the numerator is not less than that of the denominator, we find by actual division

$$\frac{x^4-6x^2+16}{(x+2)(x^2-4)}=x-2+\frac{2x^2}{(x+2)(x^2-4)}. \tag{1}$$

We now assume $\frac{2x^2}{(x+2)(x^2-4)}=\frac{A}{(x+2)^2}+\frac{B}{x+2}+\frac{C}{x-2}$. (2)

Clearing of fractions, we have

$$\begin{aligned}2x^2&=A(x-2)+B(x^2-4)+C(x+2)^2\\&=(B+C)x^2+(A+4C)x+(-2A-4B+4C).\end{aligned} \tag{3}$$

Equating the coefficients of like powers of x, we obtain the equations

$$\begin{aligned}B+C&=2,\\A+4C&=0,\\-2A-4B+4C&=0,\end{aligned}$$

whence $A=-2$, $B=\frac{3}{2}$, $C=\frac{1}{2}$.

Therefore substituting in (2), we have

$$\frac{2x^2}{(x+2)(x^2-4)}=-\frac{2}{(x+2)^2}+\frac{\frac{3}{2}}{x+2}+\frac{\frac{1}{2}}{x-2},$$

so that finally

$$\frac{x^4-6x^2+16}{(x+2)(x^2-4)}=x-2-\frac{2}{(x+2)^2}+\frac{3}{2(x+2)}+\frac{1}{2(x-2)}.$$

Ex. 2. Separate into partial fractions $\frac{3x^7+x^6-6x^4-8x^2-11x+9}{2(x^3-1)^2}$.

By division we first find

$$\frac{3x^7+x^6-6x^4-8x^2-11x+9}{2(x^3-1)^2}=\frac{3}{2}x+\frac{1}{2}+\frac{x^3-4x^2-7x+4}{(x^3-1)^2}. \tag{1}$$

We now assume

$$\begin{aligned}\frac{x^3-4x^2-7x+4}{(x^3-1)^2}=\frac{A}{(x-1)^2}+\frac{B}{x-1}+\frac{Cx+D}{(x^2+x+1)^2}\\+\frac{Ex+F}{x^2+x+1},\end{aligned} \tag{2}$$

and clear of fractions. The result is

$$\begin{aligned}x^3-4x^2-7x+4=(B+E)x^5+(A+B-E+F)x^4\\+(2A+B+C-F)x^3+(3A-B-2C+D-E)x^2\\+(2A-B+C-2D+E-F)x+(A-B+D+F).\end{aligned} \tag{3}$$

Equating the coefficients of like powers of x in (3), we obtain the equations

$$\begin{aligned} B+E&=0,\\ A+B-E+F&=0,\\ 2A+B+C-F&=1,\\ 3A-B-2C+D-E&=-4,\\ 2A-B+C-2D+E-F&=-7,\\ A-B+D+F&=4, \end{aligned} \tag{4}$$

whence $A=-\frac{2}{3}$, $B=0$, $C=3$, $D=4$, $E=0$, $F=\frac{2}{3}$.

Substituting these values in (2), we have

$$\frac{x^3-4x^2-7x+4}{(x^3-1)^2}=-\frac{2}{3(x-1)^2}+\frac{3x+4}{(x^2+x+1)^2}+\frac{2}{3(x^2+x+1)},$$

so that finally

$$\frac{3x^7+x^6-6x^4-8x^2-11x+9}{2(x^3-1)^2}=\frac{3}{2}x+\frac{1}{2}-\frac{2}{3(x-1)^2}+\frac{3x+4}{(x^2+x+1)^2}+\frac{2}{3(x^2+x+1)}.$$

55. Proof of the possibility of separation into partial fractions. In the last two articles we have assumed that the given fraction can be separated into partial fractions, and proceeding on this assumption we have been able to determine the unknown constants which were assumed in the numerators. We will now give a proof that a fraction can always be broken up into partial fractions of the types assumed in §§ 53, 54.

Let the given fraction be $\frac{f(x)}{F(x)}$, where $f(x)$ and $F(x)$ are polynomials having no common factor.

Let $x-r$ be a linear factor of $F(x)$ which occurs m times, and $F_1(x)$ be the product of the remaining factors. Then $F(x)=(x-r)^mF_1(x)$ and

$$\frac{f(x)}{F(x)}=\frac{f(x)}{(x-r)^mF_1(x)}. \tag{1}$$

Now the equation

$$\frac{f(x)}{(x-r)^mF_1(x)}=\frac{A}{(x-r)^m}+\frac{f(x)-AF_1(x)}{(x-r)^mF_1(x)} \tag{2}$$

is identically true, A being any constant.

If we can determine A so that

$$f(r)-AF_1(r)=0, \tag{3}$$

then $f(x)-AF_1(x)$ is divisible by $x-r$ (I, § 40) and may be denoted by $(x-r)f_1(x)$.

But by hypothesis neither $f(x)$ nor $F_1(x)$ is divisible by $(x-r)$, and hence $f(r)\neq 0$ and $F_1(r)\neq 0$. Therefore, from (3),

$$A=\frac{f(r)}{F_1(r)}, \tag{4}$$

a constant, which is not zero.

With this value of A we have

$$\frac{f(x)}{F(x)}=\frac{A}{(x-r)^m}+\frac{f_1(x)}{(x-r)^{m-1}F_1(x)}. \tag{5}$$

Applying this same method to $\dfrac{f_1(x)}{(x-r)^{m-1}F_1(x)}$, we have

$$\frac{f_1(x)}{(x-r)^{m-1}F_1(x)}=\frac{A_1}{(x-r)^{m-1}}+\frac{f_2(x)}{(x-r)^{m-2}F_1(x)},$$

where $A_1=\dfrac{f_1(r)}{F_1(r)}$ and $(x-r)f_2(x)=f_1(x)-A_1F_1(x)$.

It is to be noted, however, that A_1 may be zero, since $f_1(r)$ may be zero; but A_1 cannot be infinite since $F_1(r)\neq 0$.

Applying this method m times in succession, we have

$$\frac{f(x)}{F(x)}=\frac{A}{(x-r)^m}+\frac{A_1}{(x-r)^{m-1}}+\frac{A_2}{(x-r)^{m-2}}+\cdots+\frac{A_m}{x-r}+\frac{f_m(x)}{F_1(x)},$$

where A, A_1, A_2, $\cdots$ A_m are all finite constants, of which A is the only one which cannot be zero.

By the above reasoning it is evident that corresponding to any linear factor of the denominator which occurs m times we may assume m fractions, the numerators of which are constant, and the denominators of which are respectively the mth, the $(m-1)$st, $\cdots$, 1st powers of the factor.

After these fractions have been removed, the remaining fraction, i.e. $\dfrac{f_m(x)}{F_1(x)}$, may be treated in the same way.

In the above discussion r and the coefficients of $f(x)$ and $F(x)$ may be real or complex. Consequently, the method may be applied successively to each factor of $F(x)$, thus making a complete separation into partial fractions. If, however, $f(x)$ and $F(x)$ have real coefficients and we wish to confine ourselves to real polynomials, we will apply the method to real linear factors only, and proceed in the next article to deal with the quadratic factors.

56. Proceeding now to the case of a quadratic factor of $F(x)$, of the form $(x-a)^2+b^2$, which cannot be separated into real linear factors, let

$$F(x)=[(x-a)^2+b^2]^m F_1(x).$$

Then
$$\frac{f(x)}{F(x)}=\frac{f(x)}{[(x-a)^2+b^2]^m F_1(x)}. \tag{1}$$

Now the equation

$$\frac{f(x)}{[(x-a)^2+b^2]^m F_1(x)}=\frac{Ax+B}{[(x-a)^2+b^2]^m}+\frac{f(x)-(Ax+B)F_1(x)}{[(x-a)^2+b^2]^m F_1(x)} \tag{2}$$

is identically true, A and B being any constants.

If we can determine A and B so that

$$\begin{aligned} &f(a+bi)-[A(a+bi)+B]F_1(a+bi)=0,\\ \text{and}\qquad &f(a-bi)-[A(a-bi)+B]F_1(a-bi)=0, \end{aligned} \tag{3}$$

then $f(x)-(Ax+B)F_1(x)$ is divisible by $x-a-bi$ and $x-a+bi$ (I, § 40), and hence is divisible by their product $(x-a)^2+b^2$, and we can place

$$f(x)-(Ax+B)F_1(x)=[(x-a)^2+b^2]f_1(x).$$

By hypothesis neither $f(x)$ nor $F_1(x)$ is divisible by $(x-a)^2+b^2$; hence $f(a\pm bi)\neq 0$ and $F_1(a\pm bi)\neq 0$.

Denoting $\dfrac{f(a+bi)}{F_1(a+bi)}$ by $P+Qi$, and $\dfrac{f(a-bi)}{F_1(a-bi)}$ by $P-Qi$, we shall have

$$A(a+bi)+B=P+Qi$$

and
$$A(a-bi)+B=P-Qi,$$

where P and Q are finite quantities, both of which may not be zero at the same time.

Therefore
$$\begin{aligned} aA + B &= P, \\ bA &= Q, \end{aligned} \tag{4}$$

two equations from which A and B are found to have real finite values which cannot both be zero.

With these values for A and B we have

$$\frac{f(x)}{F(x)} = \frac{Ax+B}{[(x-a)^2+b^2]^m} + \frac{f_1(x)}{[(x-a)^2+b^2]^{m-1}F_1(x)},$$

and repeating this process as in the case of the linear factor we have finally

$$\frac{f(x)}{F(x)} = \frac{Ax+B}{[(x-a)^2+b^2]^m} + \frac{A_1x+B_1}{[(x-a)^2+b^2]^{m-1}} + \cdots + \frac{f_m(x)}{F_1(x)}.$$

It should be added that A and B may not be zero at the same time, and that any or all of the other constants may be zero.

The same method may evidently be applied to each one of the quadratic factors of $F(x)$.

To sum up, if

$$F(x) = (x-r_1)^m(x-r_2)^n \cdots [(x-a)^2+b^2]^l \cdots,$$

and we apply the above methods to the linear factors in succession and then to the quadratic factors in succession, we have finally

$$\begin{aligned} \frac{f(x)}{F(x)} = {} & \frac{A}{(x-r_1)^m} + \frac{A_1}{(x-r_1)^{m-1}} + \cdots + \frac{A_m}{x-r_1} \\ & + \frac{B}{(x-r_2)^n} + \frac{B_1}{(x-r_2)^{n-1}} + \cdots + \frac{B_n}{x-r_2} \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + \frac{Cx+D}{[(x-a)^2+b^2]^l} + \frac{C_1x+D_1}{[(x-a)^2+b^2]^{l-1}} + \cdots \\ & + \frac{C_lx+D_l}{(x-a)^2+b^2} + \cdots + I, \end{aligned}$$

where I is either zero or an integral expression in x.

But if the degree of $f(x)$ is less than that of $F(x)$, and we shall always reduce the fraction to this case by actual division, $\frac{f(x)}{F(x)}$ and all the fractions on the right-hand side of the equation are zero when $x=\infty$; hence I is zero and the fraction is separated into the partial fractions noted.

57. In the discussion of the last article the quadratic factors of the denominator are only those which represent the product of two conjugate imaginary factors, all the real linear factors having been previously removed. But some of these real linear factors may be surd, in which case the algebraic work of the determination of the numerators (§§ 53, 54) is burdensome. If, however, the surd factor is of the form $x-a-\sqrt{b}$, where a and b are rational, this work may be avoided in the following manner:

(1) It may be shown by a method similar to that used in I, §§ 44, 45, that if $F(x)$ has only rational coefficients, and $x-a-\sqrt{b}$ is a factor of $F(x)$, then $x-a+\sqrt{b}$ is also a factor, and hence that $F(x)$ contains $(x-a)^2-b$ as a factor.

(2) If $[(x-a)^2-b]^n$ is a factor of the denominator $F(x)$, the other factor being $F_1(x)$, then

$$F(x)=[(x-a)^2-b]^n F_1(x).$$

Then the rational fraction $\frac{f(x)}{F(x)}$ may be proved equal to

$$\frac{Cx+D}{[(x-a)^2-b]^n}+\frac{f_1(x)}{[(x-a)^2-b]^{n-1}F_1(x)}.$$

The proof, being similar to that of the last article, is left to the student.

Accordingly, if all the coefficients of the denominator are rational, and the surd factors, if any, are of the type just noted, the fraction will be separated into partial fractions, the denominators of which shall be of the forms $(x-r)^n$, $[(x-a)^2+b^2]^n$, and $[(x-a)^2-b]^n$.

58. Integration of rational fractions. The integration of a rational fraction in general consists of two steps: (1) the separation of the fraction into partial fractions; (2) the integration of each partial fraction, and the subsequent addition of the integrals.

There will then be four types of integrals to consider:

$$\int\frac{A\,dx}{x-r},\quad \int\frac{A\,dx}{(x-r)^n},\quad \int\frac{(Ax+B)\,dx}{(x-a)^2+b^2},\quad \text{and}\quad \int\frac{(Ax+B)\,dx}{[(x-a)^2+b^2]^n}.$$

The first three, however, have already been discussed. Turning then to the fourth, we may put that in the form

$$\int \frac{\frac{A}{2}[2(x-a)]+B+aA}{[(x-a)^2+b^2]^n}dx,$$

which is equal to the sum of the two integrals

$$\frac{A}{2}\int [(x-a)^2+b^2]^{-n}2(x-a)\,dx$$

and $$(B+aA)\int \frac{dx}{[(x-a)^2+b^2]^n}.$$

The first of these integrals is readily seen to be

$$\frac{A}{2(-n+1)}\cdot\frac{1}{[(x-a)^2+b^2]^{n-1}}.$$

The second may be evaluated by placing $x-a=b\tan\theta$. Then

$$\int \frac{dx}{[(x-a)^2+b^2]^n}=\frac{1}{b^{2n-1}}\int \cos^{2n-2}\theta\,d\theta.$$

When $n=2$, this integral is evaluated as in Ex. 3, § 13. The case $n>2$ rarely occurs in practice, but if it does occur, $\int \cos^{2n-2}\theta\,d\theta$ may be evaluated by methods of § 65, or the integral $\int \frac{dx}{[(x-a)^2+b^2]^n}$ may be evaluated by successive applications of the reduction formula

$$\int \frac{du}{(u^2+a^2)^n}=\frac{1}{2(n-1)a^2}\left[\frac{u}{(u^2+a^2)^{n-1}}+(2n-3)\int \frac{du}{(u^2+a^2)^{n-1}}\right],$$

which will be derived in § 73.

Ex. 1. Find the value of $\int \frac{(x^2+8x+3)\,dx}{x^3+2x^2-x-2}$.

Since $x^3+2x^2-x-2=(x-1)(x+1)(x+2)$, we assume

$$\frac{x^2+8x+3}{x^3+2x^2-x-2}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{x+2}.$$

Determining A, B, and C by the methods of the previous articles, we have

$$\frac{x^2+8x+3}{x^3+2x^2-x-2}=\frac{2}{x-1}+\frac{2}{x+1}-\frac{3}{x+2}.$$

$$\begin{aligned}\therefore \int\frac{(x^2+8x+3)\,dx}{x^3+2x^2-x-2}&=\int\left(\frac{2}{x-1}+\frac{2}{x+1}-\frac{3}{x+2}\right)dx\\&=2\int\frac{dx}{x-1}+2\int\frac{dx}{x+1}-3\int\frac{dx}{x+2}\\&=2\log(x-1)+2\log(x+1)-3\log(x+2)+C\\&=\log\frac{(x-1)^2(x+1)^2}{(x+2)^3}+C\\&=\log\frac{(x^2-1)^2}{(x+2)^3}+C.\end{aligned}$$

Ex. 2. Find the value of $\int\frac{(8x^4+2x^2+6x+18)\,dx}{8x^3+27}$.

By division, $$\frac{8x^4+2x^2+6x+18}{8x^3+27}=x+\frac{2x^2-21x+18}{8x^3+27}. \qquad (1)$$

The real factors of $8x^3+27$ are $2x+3$ and $4x^2-6x+9$. Therefore, we assume

$$\frac{2x^2-21x+18}{8x^3+27}=\frac{A}{2x+3}+\frac{Bx+C}{4x^2-6x+9}. \qquad (2)$$

Determining A, B, and C by the methods of the previous articles, we have

$$\frac{2x^2-21x+18}{8x^3+27}=\frac{2}{2x+3}-\frac{3x}{4x^2-6x+9}. \qquad (3)$$

$$\begin{aligned}\therefore \int\frac{(8x^4+2x^2+6x+18)\,dx}{8x^3+27}&=\int\left(x+\frac{2}{2x+3}-\frac{3x}{4x^2-6x+9}\right)dx\\&=\int x\,dx+\int\frac{2\,dx}{2x+3}-\int\frac{3x\,dx}{4x^2-6x+9}. \qquad (4)\end{aligned}$$

But $$\int x\,dx=\frac{1}{2}x^2, \qquad \int\frac{2\,dx}{2x+3}=\log(2x+3),$$

and $$\begin{aligned}\int\frac{3x\,dx}{4x^2-6x+9}&=\frac{3}{8}\int\frac{(8x-6)\,dx}{4x^2-6x+9}+\frac{9}{4}\int\frac{dx}{4x^2-6x+9}\\&=\frac{3}{8}\log(4x^2-6x+9)+\frac{3}{4\sqrt{3}}\tan^{-1}\frac{4x-3}{3\sqrt{3}}.\end{aligned}$$

Substituting the values of the integrals in (4), we have finally

$$\begin{aligned}&\int\frac{(8x^4+2x^2+6x+18)\,dx}{8x^3+27}\\&=\frac{1}{2}x^2+\log(2x+3)-\frac{3}{8}\log(4x^2-6x+9)-\frac{3}{4\sqrt{3}}\tan^{-1}\frac{4x-3}{3\sqrt{3}}+C\\&=\frac{1}{2}x^2+\log\frac{2x+3}{(4x^2-6x+9)^{\frac{3}{8}}}-\frac{3}{4\sqrt{3}}\tan^{-1}\frac{4x-3}{3\sqrt{3}}+C.\end{aligned}$$

Ex. 3. Find the value of $\int\frac{(3x^7+x^6-6x^4-8x^2-11x+9)\,dx}{2(x^3-1)^2}$.

The fraction being the same as that of Ex. 2, § 54, we have the first step of the work completed.

Hence

$$\int\frac{(3x^7+x^6-6x^4-8x^2-11x+9)\,dx}{2(x^3-1)^2}$$

$$=\int\frac{3}{2}x\,dx+\int\frac{1}{2}dx-\int\frac{2\,dx}{3(x-1)^2}+\int\frac{(3x+4)\,dx}{(x^2+x+1)^2}+\int\frac{2\,dx}{3(x^2+x+1)}. \qquad (1)$$

Now the first, the second, the third, and the fifth integrals are readily evaluated by previous methods, their values being respectively $\frac{3}{4}x^2$, $\frac{1}{2}x$, $\frac{2}{3(x-1)}$, and $\frac{4}{3\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}$.

There remains the fourth integral $\int\frac{(3x+4)\,dx}{(x^2+x+1)^2}$, which may be reduced to

$$\frac{3}{2}\int(x^2+x+1)^{-2}(2x+1)\,dx+\frac{5}{2}\int\frac{dx}{(x^2+x+1)^2}.$$

The first integral is $-\frac{3}{2(x^2+x+1)}$, while the second integral may be written in the form $40\int\frac{dx}{[(2x+1)^2+3]^2}$, and can be evaluated by placing $2x+1=\sqrt{3}\tan\theta$. The result will be

$$40\int\frac{dx}{[(2x+1)^2+3]^2}=\frac{20}{3\sqrt{3}}\int\cos^2\theta\,d\theta$$

$$=\frac{10}{3\sqrt{3}}\theta+\frac{5}{3\sqrt{3}}\sin 2\theta$$

$$=\frac{10}{3\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}+\frac{5}{6}\cdot\frac{2x+1}{x^2+x+1}.$$

Hence
$$\int\frac{(3x+4)\,dx}{(x^2+x+1)^2}=-\frac{3}{2(x^2+x+1)}+\frac{10}{3\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}$$
$$+\frac{5}{6}\cdot\frac{2x+1}{x^2+x+1}.$$

Finally, substituting the values of the integrals in (1) and simplifying, we have, as the value of the original integral,

$$\frac{3}{4}x^2+\frac{1}{2}x+\frac{2}{3(x-1)}+\frac{5x-2}{3(x^2+x+1)}+\frac{14}{3\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}+C.$$

PROBLEMS

Separate the following fractions into partial fractions:

1. $\dfrac{x^2+x-4}{6x^3+5x^2-6x}.$

2. $\dfrac{x^4+2x^3+x-1}{x^3+x^2-2x}.$

3. $\dfrac{3x+1}{4x^3+8x^2-x-2}.$

4. $\dfrac{x^3-11x-1}{x^3+2x^2-5x-6}.$

5. $\dfrac{3x^2-x-3}{x^4+x^3}.$

6. $\dfrac{x^2-7x+2}{4x^4-4x^3+x^2}.$

7. $\dfrac{x^3-x^2-12x-8}{x(x+2)^3}.$

8. $\dfrac{8x^4-4x^3-2x^2+7x}{(2x+1)^3}.$

9. $\dfrac{x^2-x+6}{2x^3+x^2+4x+2}.$

10. $\dfrac{3x^2-4x}{x^3-2x+4}.$

11. $\dfrac{12x^5+x^3+2x^2-x-2}{3x^4-2x^2-1}.$

12. $\dfrac{2x^3-x^2+5x+2}{x^4+3x^2+4}.$

13. $\dfrac{x^5-x^4+x^3-2x^2-2x-1}{x^4+3x^2+2}.$

14. $\dfrac{x^4+5x^2+2x+6}{x^5+6x^3+9x}.$

15. $\dfrac{x^4+x^3-4x^2-6x+8}{x^5-2x^4-4x^3+8x^2+4x-8}.$

16. $\dfrac{x^3-x^2+x+3}{(x^2+x+1)^2}.$

Find the values of the following integrals:

17. $\displaystyle\int\frac{(14x+3)\,dx}{4x^2+4x-15}.$

18. $\displaystyle\int\frac{(3x+4)\,dx}{2+3x-9x^2}.$

19. $\displaystyle\int\frac{(3x-10)\,dx}{2x^2+5x-12}.$

20. $\displaystyle\int\frac{(3x+7)\,dx}{x^2+4x+1}.$

21. $\displaystyle\int\frac{(5x+4)\,dx}{4x^2+4x+2}.$

22. $\displaystyle\int\frac{x^3+x^2+x+1}{3-2x-x^2}\,dx.$

23. $\displaystyle\int\frac{9x^3+3x^2-x-3}{9x^2+12x+8}\,dx.$

24. $\displaystyle\int\frac{x^3+2x^2+3x}{x^2-2x-1}\,dx.$

25. $\displaystyle\int\frac{2x^2-11x-6}{x^3+x^2-6x}\,dx.$

26. $\displaystyle\int\frac{14x^2-3x-15}{x^3-x}\,dx.$

27. $\displaystyle\int\frac{(6-12x)\,dx}{x^3-2x^2-5x+6}.$

28. $\displaystyle\int\frac{(3x^2-10x-16)\,dx}{x^3+x^2-4x-4}.$

29. $\displaystyle\int\frac{4x^3+x^2+1}{2x^3+5x^2-3x}\,dx.$

30. $\displaystyle\int\frac{6x^4+17x^3-3x^2-6x+4}{2x^3+7x^2+2x-3}\,dx.$

31. $\displaystyle\int\frac{4x^4+10x^3-8x^2-20x+1}{4x^3+8x^2-9x-18}\,dx.$

32. $\displaystyle\int\frac{16x^2-18x+3}{4x^3-4x^2+x}\,dx.$

33. $\displaystyle\int\frac{(x^2+2x+4)\,dx}{x^3+6x^2+12x+8}.$

34. $\displaystyle\int\frac{(6x^2+10x+9)\,dx}{4x^3+8x^2-3x-9}.$

35. $\int \frac{(3 - 2x - 4x^2)\,dx}{4x^3 + 12x^2 + 9x + 2}$.

36. $\int \frac{(5x - 2)\,dx}{x^3 + 2x^2 - 4x - 8}$.

37. $\int \frac{2 + 5x - x^2 - 2x^3}{x^4 + 2x^3 + x^2}\,dx$.

38. $\int \frac{9x^3 - 3x + 2}{9x^3 - 6x^2}\,dx$.

39. $\int \frac{x^4 - 2x^3 - x - 1}{x(x - 1)^3}\,dx$.

40. $\int \frac{16x^5 - 8x^4 - 8x^3 + 2x^2 - 1}{16x^4 - 8x^2 + 1}\,dx$.

41. $\int \frac{4x^2 + 6x - 5}{x^3 + x^2 - 2}\,dx$.

42. $\int \frac{(x^2 - 13x)\,dx}{x^3 - 3x^2 - 2x + 2}$.

43. $\int \frac{3x^3 - 12x + 8}{x^4 - 16}\,dx$.

44. $\int \frac{x\,dx}{x^4 + (a - b)x^2 - ab}$.

45. $\int \frac{(15x^2 - 29x - 17)\,dx}{(3x^2 + 2)(x^2 - 2x + 3)}$.

46. $\int \frac{(2x^2 + x + 1)\,dx}{(x^2 + 3)(2x^2 + x + 5)}$.

47. $\int \frac{x^4 - 3x^3 - 7x^2 - 4x + 2}{x^4 + 4}\,dx$.

48. $\int \frac{4x^4 - 4x^3 + x^2 + 13x - 3}{8x^3 + 27}\,dx$.

49. $\int \frac{9x^3 + 20x^2 + 21x - 5}{(x^2 - 3)(3x^2 + 3x + 5)}\,dx$.

50. $\int \frac{x^2 + 2x}{(x^2 + 3)^2}\,dx$.

51. $\int \frac{2x^3 + 10x^2 + 4x}{(2x^2 + 1)^2}\,dx$.

52. $\int \frac{5x^3 - 11x^2 + 19x - 27}{x^5 + 6x^3 + 9x}\,dx$.

53. $\int \frac{2 + 10x + 2x^2 - 6x^3}{x - 4x^3 + 4x^5}\,dx$.

CHAPTER VII

SPECIAL METHODS OF INTEGRATION

59. Rationalization. By a suitable substitution of a new variable an irrational function may sometimes be made a rational function of the new variable. In this case the integrand is said to be *rationalized,* and the integration is performed by the methods of Chap. VI. We shall now discuss in §§ 60–63 some of the cases in which this method is possible, together with the appropriate substitution in each.

60. Integrand containing fractional powers of $a + bx$. Expressions involving fractional powers of $a + bx$ and integral powers of x can be rationalized by assuming

$$a + bx = z^n,$$

where n is the least common denominator of the fractional exponents of the binomial.

For if $a + bx = z^n$, then $x = \frac{1}{b}(z^n - a)$ and $dx = \frac{n}{b}z^{n-1}dz$. Also, if $(a + bx)^p$ is one of the fractional powers of $a + bx$, $(a + bx)^p = z^{pn}$, where pn is an integer. Since x, dx, and the fractional powers of $a + bx$ can all be expressed rationally in terms of z, it follows that the integrand will be a rational function of z when the substitution has been completed.

Ex. 1. Find the value of $\int \frac{x^2\,dx}{(1 + 2x)^{\frac{1}{3}}}$.

Here we let $1 + 2x = z^3$; then $x = \frac{1}{2}(z^3 - 1)$, and $dx = \frac{3}{2}z^2\,dz$.

Therefore
$$\int \frac{x^2\,dx}{(1 + 2x)^{\frac{1}{3}}} = \frac{3}{8}\int (z^7 - 2z^4 + z)\,dz$$
$$= \tfrac{3}{8}\left(\tfrac{1}{8}z^8 - \tfrac{2}{5}z^5 + \tfrac{1}{2}z^2\right) + C$$
$$= \tfrac{3}{320}z^2(5z^6 - 16z^3 + 20) + C.$$

Replacing z by its value $(1 + 2x)^{\frac{1}{3}}$ and simplifying, we have

$$\int \frac{x^2\,dx}{(1 + 2x)^{\frac{1}{3}}} = \frac{3}{320}(1 + 2x)^{\frac{2}{3}}(9 - 12x + 20x^2) + C.$$

Ex. 2. Find the value of $\int \frac{(x+2)^{\frac{1}{2}} - (x+2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{2}} + 2}\,dx$.

Since the least common denominator of the exponents of the binomial is 4, we assume $x + 2 = z^4$; then $x = z^4 - 2$, and $dx = 4z^3\,dz$. On substitution, the integral becomes

$$4\int \frac{z^5 - z^4}{z^2+2}\,dz = 4\int \left(z^3 - z^2 - 2z + 2 + \frac{4z-4}{z^2+2}\right)dz$$
$$= 4\left[\frac{1}{4}z^4 - \frac{1}{3}z^3 - z^2 + 2z + 2\log(z^2+2) - \frac{4}{\sqrt{2}}\tan^{-1}\frac{z}{\sqrt{2}}\right] + C.$$

Replacing z by its value $(x+2)^{\frac{1}{4}}$, we have

$$\int \frac{(x+2)^{\frac{1}{2}} - (x+2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{2}} + 2}\,dx$$
$$= x + 2 - \tfrac{4}{3}(x+2)^{\frac{3}{4}} - 4(x+2)^{\frac{1}{2}} + 8(x+2)^{\frac{1}{4}} + 8\log\left(\sqrt{x+2} + 2\right)$$
$$- 8\sqrt{2}\tan^{-1}\frac{(x+2)^{\frac{1}{4}}}{\sqrt{2}} + C.$$

61. Integrand containing fractional powers of $a + bx^n$. If the integrand is the product

$$x^m(a + bx^n)^{\frac{q}{r}},$$

where q and r are integers, there are two cases in which rationalization is possible.

Case I. *When* $\frac{m+1}{n}$ *is an integer or zero.* Let us assume

$$a + bx^n = z^r$$

and observe the result of the substitution. Then

$$x = \frac{1}{b^{\frac{1}{n}}}(z^r - a)^{\frac{1}{n}}, \quad \text{and} \quad dx = \frac{r}{nb^{\frac{1}{n}}}(z^r - a)^{\frac{1}{n}-1}z^{r-1}\,dz.$$

Therefore

$$x^m(a + bx^n)^{\frac{q}{r}}dx = \frac{r}{nb^{\frac{m+1}{n}}}z^{q+r-1}(z^r - a)^{\frac{m+1}{n}-1}\,dz.$$

But if $\frac{m+1}{n}$ is an integer or zero, this new integrand is a rational function of z, and the assumed substitution is an effective one.

Ex. 1. Find the value of $\int x^5(1+2x^3)^{\frac{1}{2}}dx$.

Since $\frac{m+1}{n}=2$, we assume $1+2x^3=z^2$, whence $x^3=\frac{1}{2}(z^2-1)$, and $x^2dx=\frac{1}{3}z\,dz$. The new integral is $\frac{1}{6}\int(z^4-z^2)\,dz$, which reduces to $\frac{1}{90}z^3(3z^2-5)+C$.

Replacing z by its value, we have

$$\int x^5(1+2x^3)^{\frac{1}{2}}\,dx=\tfrac{1}{45}(1+2x^3)^{\frac{3}{2}}(3x^3-1)+C.$$

Case II. *When* $\frac{m+1}{n}+\frac{q}{r}$ *is an integer or zero.* Here we will assume

$$a+bx^n=x^nz^r.$$

Then

$$x=\frac{a^{\frac{1}{n}}}{(z^r-b)^{\frac{1}{n}}}\quad\text{and}\quad dx=-\frac{a^{\frac{1}{n}}r\,z^{r-1}}{n(z^r-b)^{\frac{1}{n}+1}}\,dz.$$

Therefore

$$x^m(a+bx^n)^{\frac{q}{r}}dx=-\frac{ra^{\frac{m+1}{n}+\frac{q}{r}}z^{q+r-1}}{n(z^r-b)^{\frac{m+1}{n}+\frac{q}{r}+1}}\,dz.$$

This new expression is a rational function of z.

Ex. 2. Find the value of $\int\frac{(2+x^2)^{\frac{3}{2}}}{x^2}dx$.

Here $\frac{m+1}{n}+\frac{q}{r}=1$, and accordingly we let $2+x^2=x^2z^2$. After the substitution we have, as the new integral, $-2\int\frac{z^4\,dz}{(z^2-1)^2}$, the value of which is $\frac{z(3-2z^2)}{z^2-1}+\frac{3}{2}\log\frac{z+1}{z-1}+C$.

Replacing z by its value, we have

$$\int\frac{(2+x^2)^{\frac{3}{2}}}{x^2}dx=\frac{\sqrt{2+x^2}(x^2-4)}{2x}+\frac{3}{2}\log\frac{\sqrt{2+x^2}+x}{\sqrt{2+x^2}-x}+C.$$

62. Integrand containing integral powers of $\sqrt{a+bx+x^2}$. If the integrand contains only integral powers of x and of $\sqrt{a+bx+x^2}$, it may be rationalized by the substitution

$$\sqrt{a+bx+x^2}=z-x;$$

for from this equation

$$x=\frac{z^2-a}{2z+b},\quad\sqrt{a+bx+x^2}=\frac{z^2+bz+a}{2z+b},$$

and

$$dx=\frac{2(z^2+bz+a)}{(2z+b)^2}\,dz.$$

The result of the substitution is evidently a rational function of z.

Ex. Find the value of $\int\frac{x^2\,dx}{\sqrt{1+x+x^2}}$.

Letting $\sqrt{1+x+x^2}=z-x$, we have as the new integral $2\int\frac{(z^2-1)^2}{(2z+1)^3}dz$, which, by the method of Chap. VI, is

$$\frac{1}{8}\left[z^2-3z-\frac{48z+33}{4(2z+1)^2}-\log(2z+1)\right]+C.$$

Replacing z by its value $x+\sqrt{1+x+x^2}$, we have as the value of the original integral

$$\frac{1}{8}\left[(x+\sqrt{1+x+x^2})^2-3(x+\sqrt{1+x+x^2})\right.$$
$$\left.-\frac{48x+33+48\sqrt{1+x+x^2}}{4(2x+1+2\sqrt{1+x+x^2})^2}-\log(2x+1+2\sqrt{1+x+x^2})\right]+C.$$

If the coefficient of x^2 under the radical sign is any positive constant other than unity, we may factor it out, thereby bringing the expression under the case just discussed.

63. Integrand containing integral powers of $\sqrt{a+bx-x^2}$. In this case the substitution of the previous article fails, as x could not be expressed rationally in terms of z. We may now, however, write

$$a+bx-x^2=a-(x^2-bx)=\frac{4a+b^2}{4}-\left(x-\frac{b}{2}\right)^2,$$

an expression which can be factored into two linear factors of the form $(r_1+x)(r_2-x)$, which are real except when $\frac{4a+b^2}{4}\leqq 0$. But then it is evident that $\frac{4a+b^2}{4}-\left(x-\frac{b}{2}\right)^2$ is negative for all values of x, and hence $\sqrt{a+bx-x^2}$ is always imaginary.

Now place

$$\sqrt{a+bx-x^2}=\sqrt{(r_1+x)(r_2-x)}=z(r_2-x).$$

Solving this equation for x, we have

$$x=\frac{r_2z^2-r_1}{z^2+1},\quad \sqrt{a+bx-x^2}=\frac{(r_1+r_2)z}{z^2+1},$$

and

$$dx=\frac{2(r_1+r_2)z}{(z^2+1)^2}dz.$$

After the substitution the new integrand is evidently a rational function of z.

Ex. Find the value of $\int \frac{dx}{x\sqrt{2-x-x^2}}$.

Since $2 - x - x^2 = (2 + x)(1 - x)$, we assume

$$\sqrt{2-x-x^2} = \sqrt{(2+x)(1-x)} = z(1-x).$$

The result of the substitution is the integral $2\int \frac{dz}{z^2-2}$, which is equal to $\frac{1}{\sqrt{2}} \log \frac{z-\sqrt{2}}{z+\sqrt{2}} + C$.

Replacing z by its value $\sqrt{\frac{2+x}{1-x}}$, we have

$$\int \frac{dx}{x\sqrt{2-x-x^2}} = \frac{1}{\sqrt{2}} \log \frac{\sqrt{2+x}-\sqrt{2-2x}}{\sqrt{2+x}+\sqrt{2-2x}} + C.$$

64. Integration of trigonometric functions. There are certain types of trigonometric functions for which definite rules of procedure may be stated. The simplest case is that in which the integrand is any power of a trigonometric function multiplied by its differential, e.g. $\int \sin^{\frac{3}{2}} x \cos x\, dx$, which may be integrated by one of the fundamental formulas. It is evident that this class requires no further consideration here. To deal with the more complex cases it is usually necessary to make a trigonometric transformation of the integrand, the choice of the particular transformation being guided by the formulas for the differentials of the trigonometric functions. Some of these transformations are given in §§ 65–70.

65. Integrals of the forms $\int \sin^n x\, dx$ and $\int \cos^n x\, dx$. We may distinguish three cases.

CASE I. *n an odd integer.* In the integral $\int \sin^n x\, dx$ we may place

$$\sin^n x\, dx = \sin^{n-1} x \sin x\, dx.$$

Now $\sin x\, dx = -d(\cos x)$, and $\sin^{n-1} x = (1-\cos^2 x)^{\frac{n-1}{2}}$, which is a rational function of $\cos x$ since $\frac{n-1}{2}$ is an integer.

Then
$$\int \sin^n x\, dx = -\int (1-\cos^2 x)^{\frac{n-1}{2}} d(\cos x).$$

In like manner we may prove

$$\int \cos^n x\, dx = \int (1-\sin^2 x)^{\frac{n-1}{2}} d(\sin x).$$

Ex. 1. Find the value of $\int \sin^5 x\,dx$.

$$\int \sin^5 x\,dx = -\int (1-\cos^2 x)^2\,d(\cos x)$$
$$= -\cos x + \tfrac{2}{3}\cos^3 x - \tfrac{1}{5}\cos^5 x + C.$$

Ex. 2. Find the value of $\int \frac{dx}{\cos^3 x}$.

$\int \frac{dx}{\cos^3 x} = \int \frac{d(\sin x)}{(1-\sin^2 x)^2}$. To integrate, place $\sin x = z$. Then

$$\int \frac{d(\sin x)}{(1-\sin^2 x)^2} = \int \frac{dz}{(1-z^2)^2} = \frac{1}{2}\frac{z}{1-z^2} - \frac{1}{4}\log\frac{1-z}{1+z} + C$$
$$= \frac{1}{2}\frac{\sin x}{\cos^2 x} - \frac{1}{4}\log\frac{1-\sin x}{1+\sin x} + C.$$

CASE II. *n a positive even integer.* In this case we may evaluate the integral by transforming the integrand by the trigonometric formulas

$$\sin^2 x = \tfrac{1}{2}(1-\cos 2x),$$
$$\cos^2 x = \tfrac{1}{2}(1+\cos 2x),$$

as shown in the following example.

Ex. 3. Find the value of $\int \cos^4 x\,dx$.

Applying the second formula above, we have

$$\int \cos^4 x\,dx = \tfrac{1}{4}\int dx + \tfrac{1}{2}\int \cos 2x\,dx + \tfrac{1}{4}\int \cos^2 2x\,dx.$$

Applying this formula a second time, we have

$$\int \cos^2 2x\,dx = \tfrac{1}{2}\int (1+\cos 4x)\,dx.$$

Completing all the integrations, we have finally

$$\tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C.$$

CASE III. *n a negative even integer.* In this case we replace $\sin x$ by $\frac{1}{\csc x}$ and $\cos x$ by $\frac{1}{\sec x}$ and proceed as in § 68.

66. Integrals of the form $\int \sin^m x \cos^n x\,dx$. There are two cases in which an integral of this type may be readily evaluated.

CASE I. *Either m or n a positive odd integer.* If m, for example, is a positive odd integer, we place

$$\sin^m x \cos^n x\,dx = \sin^{m-1} x \cos^n x(\sin x\,dx)$$
$$= -(1-\cos^2 x)^{\frac{m-1}{2}}\cos^n x\,d(\cos x).$$

Since by the hypothesis $\frac{m-1}{2}$ is a positive integer, it is evident that the new integral can readily be evaluated.

Similarly, if n is an odd integer we express the integrand as a function of $\sin x$.

Ex. 1. Find the value of $\int\sqrt{\sin x}\cos^3 x\,dx$.

$$\int\sqrt{\sin x}\cos^3 x\,dx = \int\sqrt{\sin x}(1-\sin^2 x)\,d(\sin x)$$
$$= \tfrac{2}{21}\sin^{\frac{3}{2}}x(7-3\sin^2 x)+C.$$

Sometimes when one of the exponents is a negative integer the same method is applicable, but it is apt to lead to functions the integration of which is laborious.

Ex. 2. Find the value of $\int\frac{\sin^2 x}{\cos x}dx$.

$$\int\frac{\sin^2 x}{\cos x}dx = \int\frac{\sin^2 x\,d(\sin x)}{1-\sin^2 x} = \int\frac{z^2dz}{1-z^2} \qquad \text{(where } z = \sin x\text{)}$$
$$= -z+\frac{1}{2}\log\frac{1+z}{1-z}+C$$
$$= -\sin x+\frac{1}{2}\log\frac{1+\sin x}{1-\sin x}+C.$$

Case II. *Both m and n positive even integers.* In this case the integrand is transformed into functions of $2x$ by the two formulas of the last article and the additional formula

$$\sin x\cos x = \tfrac{1}{2}\sin 2x.$$

Ex. 3. Find the value of $\int\sin^2 x\cos^4 x\,dx$.

Placing $\qquad \sin^2 x\cos^4 x = (\sin x\cos x)^2\cos^2 x,$

we have $\qquad \sin^2 x\cos^4 x = \tfrac{1}{8}\sin^2 2x(1+\cos 2x).$

Therefore $\qquad \int\sin^2 x\cos^4 x\,dx = \tfrac{1}{8}\int\sin^2 2x\,dx + \tfrac{1}{8}\int\sin^2 2x\cos 2x\,dx.$

Applying the same method again, we have

$$\int\sin^2 2x\,dx = \tfrac{1}{2}\int(1-\cos 4x)\,dx.$$

Completing all the integrations, we have finally

$$\int\sin^2 x\cos^4 x\,dx = \tfrac{1}{16}x+\tfrac{1}{48}\sin^3 2x-\tfrac{1}{64}\sin 4x+C.$$

67. Integrals of the forms $\int \tan^n x\, dx$ and $\int \operatorname{ctn}^n x\, dx$. Since $\tan x$ and $\operatorname{ctn} x$ are reciprocals of each other, we need consider only the case in which n is a positive quantity. Accordingly, *if n is a positive integer* we may proceed as follows. Placing

$$\tan^n x = \tan^{n-2} x \tan^2 x,$$

and substituting for $\tan^2 x$ its value in terms of $\sec x$, we have

$$\tan^n x = \tan^{n-2} x (\sec^2 x - 1).$$

Therefore $$\int \tan^n x\, dx = \int \tan^{n-2} x \sec^2 x\, dx - \int \tan^{n-2} x\, dx$$
$$= \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x\, dx.$$

It is evident that the original integral may be completely evaluated by successive applications of this method. The same method is evidently applicable to the integral $\int \operatorname{ctn}^n x\, dx$.

Ex. 1. Find the value of $\int \tan^5 x\, dx$.

Placing $$\tan^5 x = \tan^3 x \tan^2 x = \tan^3 x (\sec^2 x - 1),$$

we have $$\int \tan^5 x\, dx = \int \tan^3 x \sec^2 x\, dx - \int \tan^3 x\, dx$$
$$= \tfrac{1}{4} \tan^4 x - \int \tan^3 x\, dx.$$

Again, placing $$\tan^3 x = \tan x (\sec^2 x - 1),$$

we have $$\int \tan^3 x\, dx = \int \tan x \sec^2 x\, dx - \int \tan x\, dx$$
$$= \tfrac{1}{2} \tan^2 x + \log \cos x + C.$$

Hence, by substitution,

$$\int \tan^5 x\, dx = \tfrac{1}{4} \tan^4 x - \tfrac{1}{2} \tan^2 x - \log \cos x + C.$$

Ex. 2. Find the value of $\int \operatorname{ctn}^4 2x\, dx$.

Placing $$\operatorname{ctn}^4 2x = \operatorname{ctn}^2 2x (\csc^2 2x - 1),$$

we have $$\int \operatorname{ctn}^4 2x\, dx = \int \operatorname{ctn}^2 2x \csc^2 2x\, dx - \int \operatorname{ctn}^2 2x\, dx$$
$$= -\tfrac{1}{6} \operatorname{ctn}^3 2x - \int \operatorname{ctn}^2 2x\, dx.$$

Again, placing $$\operatorname{ctn}^2 2x = \csc^2 2x - 1,$$

we have $$\int \operatorname{ctn}^2 2x\, dx = \int (\csc^2 2x - 1)\, dx$$
$$= -\tfrac{1}{2} \operatorname{ctn} 2x - x + C.$$

Hence $$\int \operatorname{ctn}^4 2x\, dx = -\tfrac{1}{6} \operatorname{ctn}^3 2x + \tfrac{1}{2} \operatorname{ctn} 2x + x + C.$$

68. Integrals of the forms $\int \sec^n x\,dx$ and $\int \csc^n x\,dx$. We shall consider these integrals only for the case in which n is an integer.

CASE I. *If n is a positive even integer*, we place

$$\begin{aligned}\int \sec^n x\,dx &= \int \sec^{n-2} x \sec^2 x\,dx\\ &= \int (1+\tan^2 x)^{\frac{n-2}{2}}\,d(\tan x),\end{aligned}$$

where the integrand is a rational function of $\tan x$, since $\frac{n-2}{2}$ is a positive integer.

In the same manner, we may show that

$$\int \csc^n x\,dx = -\int (1+\operatorname{ctn}^2 x)^{\frac{n-2}{2}}\,d(\operatorname{ctn} x).$$

Ex. 1. Find the value of $\int \sec^4 3x\,dx$.

$$\begin{aligned}\int \sec^4 3x\,dx &= \tfrac{1}{3}\int (1+\tan^2 3x)\,d(\tan 3x)\\ &= \tfrac{1}{9}\tan 3x\,(3+\tan^2 3x)+C.\end{aligned}$$

CASE II. *If n is any integer other than a positive even integer*, it is evident that the integral falls under the case of § 65 when $\sec x$ is replaced by $\frac{1}{\cos x}$ or when $\csc x$ is replaced by $\frac{1}{\sin x}$, as the case may be.

Ex. 2. Find the value of $\int \sec x\,dx$.

$$\begin{aligned}\int \sec x\,dx &= \int \frac{dx}{\cos x} = \int \frac{\cos x\,dx}{\cos^2 x}\\ &= -\int \frac{d(\sin x)}{\sin^2 x - 1} = -\frac{1}{2}\log\frac{1-\sin x}{1+\sin x}+C\\ &= \frac{1}{2}\log\frac{1+\sin x}{1-\sin x}+C\\ &= \frac{1}{2}\log\frac{(1+\sin x)^2}{1-\sin^2 x}+C\\ &= \frac{1}{2}\log\left(\frac{1+\sin x}{\cos x}\right)^2+C\\ &= \log(\sec x+\tan x)+C.\end{aligned}$$

In like manner it may be shown that $\int \csc x\,dx = \log(\csc x - \operatorname{ctn} x) + C$.

69. Integrals of the forms $\int \tan^m x \sec^n x\,dx$ and $\int \operatorname{ctn}^m x \csc^n x\,dx$.

Case I. *If n is a positive even integer,* we may write

$$\int \tan^m x \sec^n x\,dx = \int \tan^m x\,(1+\tan^2 x)^{\frac{n-2}{2}}\,d(\tan x),$$

and
$$\int \operatorname{ctn}^m x \csc^n x\,dx = -\int \operatorname{ctn}^m x\,(1+\operatorname{ctn}^2 x)^{\frac{n-2}{2}}\,d(\operatorname{ctn} x),$$

where $\dfrac{n-2}{2}$ is a positive integer.

Ex. 1. Find the value of $\int \tan^{\frac{3}{2}} 2x \sec^4 2x\,dx$.

Placing $\sec^4 2x = \sec^2 2x \sec^2 2x$, we have

$$\begin{aligned}\int \tan^{\frac{3}{2}} 2x \sec^4 2x\,dx &= \tfrac{1}{2}\int \tan^{\frac{3}{2}} 2x\,(1+\tan^2 2x)\,d(\tan 2x)\\ &= \tfrac{1}{5}\tan^{\frac{5}{2}} 2x + \tfrac{1}{9}\tan^{\frac{9}{2}} 2x + C.\end{aligned}$$

Case II. *If m is a positive odd integer,* we place

$$\begin{aligned}\int \tan^m x \sec^n x\,dx &= \int (\tan^{m-1} x \sec^{n-1} x)(\tan x \sec x\,dx)\\ &= \int \sec^{n-1} x\,(\sec^2 x - 1)^{\frac{m-1}{2}}\,d(\sec x),\end{aligned}$$

and
$$\begin{aligned}\int \operatorname{ctn}^m x \csc^n x\,dx &= \int (\operatorname{ctn}^{m-1} x \csc^{n-1} x)(\operatorname{ctn} x \csc x\,dx)\\ &= -\int \csc^{n-1} x\,(\csc^2 x - 1)^{\frac{m-1}{2}}\,d(\csc x),\end{aligned}$$

where $\dfrac{m-1}{2}$ is a positive integer.

Ex. 2. Find the value of $\int \tan^3 3x \sec^{\frac{1}{3}} 3x\,dx$.

$$\begin{aligned}\int \tan^3 3x \sec^{\frac{1}{3}} 3x\,dx &= \int (\tan^2 3x \sec^{-\frac{2}{3}} 3x)(\tan 3x \sec 3x\,dx)\\ &= \tfrac{1}{3}\int \sec^{-\frac{2}{3}} 3x\,(\sec^2 3x - 1)\,d(\sec 3x)\\ &= \tfrac{1}{7}\sec^{\frac{7}{3}} 3x - \sec^{\frac{1}{3}} 3x + C.\end{aligned}$$

Case III. *If m is an even integer and n is an odd integer,* the integral may be thrown under the cases of § 66 by placing $\sec x = \dfrac{1}{\cos x}$ and $\tan x = \dfrac{\sin x}{\cos x}$.

70. The substitution $\tan\frac{x}{2}=z$. The substitution $\tan\frac{x}{2}=z$, or $x=2\tan^{-1}z$, is of considerable value in the integration of trigonometric functions, since if the integrand involves only integral powers of the trigonometric functions, the result of this substitution is a rational function of z. For, if

$$\tan\frac{x}{2}=z,$$

then

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}=\frac{2z}{1+z^2},$$

$$\cos x = 2\cos^2\frac{x}{2}-1=\frac{1-z^2}{1+z^2},$$

$$\tan x=\frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}=\frac{2z}{1-z^2},$$

and

$$dx=\frac{2\,dz}{1+z^2}.$$

When these values for $\sin x$, $\cos x$, $\tan x$, and dx are substituted, it is evident, as stated above, that the result is a rational function.

Ex. Find the value of $\int\frac{dx}{1+2\cos x}$.

Placing $\tan\frac{x}{2}=z$, or $x=2\tan^{-1}z$, we have $\cos x=\frac{1-z^2}{1+z^2}$, and $dx=\frac{2\,dz}{1+z^2}$.

Therefore

$$\int\frac{dx}{1+2\cos x}=-2\int\frac{dz}{z^2-3}$$

$$=-\frac{2}{2\sqrt{3}}\log\frac{z-\sqrt{3}}{z+\sqrt{3}}+C$$

$$=\frac{1}{\sqrt{3}}\log\frac{\tan\frac{x}{2}+\sqrt{3}}{\tan\frac{x}{2}-\sqrt{3}}+C.$$

This method is applicable to an integral of any one of the three types $\int\frac{dx}{a+b\cos x}$, $\int\frac{dx}{a+b\sin x}$, and $\int\frac{dx}{a\cos x+b\sin x}$.

71. Algebraic reduction formulas. It was shown in § 61 that an integral of the form $\int x^m(a+bx^n)^p dx$, where p is a rational number, can be rationalized if $\frac{m+1}{n}$ or $\frac{m+1}{n}+p$ is zero or an integer. In general, however, an integral of this type is evaluated by use of the so-called *reduction formulas*, by means of which the original integral is made to depend upon another integral in which the power either of x or of the binomial $a+bx^n$ is increased or decreased.

The four reduction formulas are:

$$\int x^m(a+bx^n)^p dx$$
$$=\frac{x^{m-n+1}(a+bx^n)^{p+1}}{(np+m+1)b}-\frac{(m-n+1)a}{(np+m+1)b}\int x^{m-n}(a+bx^n)^p dx, \quad (1)$$

$$\int x^m(a+bx^n)^p dx$$
$$=\frac{x^{m+1}(a+bx^n)^p}{np+m+1}+\frac{npa}{np+m+1}\int x^m(a+bx^n)^{p-1}dx, \quad (2)$$

$$\int x^m(a+bx^n)^p dx$$
$$=\frac{x^{m+1}(a+bx^n)^{p+1}}{(m+1)a}-\frac{(np+n+m+1)b}{(m+1)a}\int x^{m+n}(a+bx^n)^p dx, \quad (3)$$

$$\int x^m(a+bx^n)^p dx$$
$$=-\frac{x^{m+1}(a+bx^n)^{p+1}}{n(p+1)a}+\frac{np+n+m+1}{n(p+1)a}\int x^m(a+bx^n)^{p+1}dx. \quad (4)$$

These formulas may be verified by differentiation; their derivation will be given in the next article.

Formulas (1) and (2) fail if $np+m+1=0$; but in that case we proved in § 61 that the integrand can be rationalized.

Formula (3) fails if $m+1=0$; and in that case also the integrand can be rationalized.

Formula (4) fails if $p+1=0$; and in that case it is evident that the integration may be performed by the method of § 60.

72. Proof of reduction formulas. To derive formula (1), § 71, we note that $d(a+bx^n) = nbx^{n-1}dx$, and accordingly place

$$x^m(a+bx^n)^p\,dx = \frac{x^{m-n+1}}{nb}(a+bx^n)^p\,nbx^{n-1}dx$$

and integrate by parts, letting

$$\frac{x^{m-n+1}}{nb} = u, \quad \text{and} \quad (a+bx^n)^p\,nbx^{n-1}dx = dv.$$

Then $\quad du = \dfrac{m-n+1}{nb}x^{m-n}dx, \quad$ and $\quad v = \dfrac{(a+bx^n)^{p+1}}{p+1}.$

As a result,

$$\int x^m(a+bx^n)^p\,dx = \frac{x^{m-n+1}(a+bx^n)^{p+1}}{nb(p+1)} - \frac{m-n+1}{nb(p+1)}\int x^{m-n}(a+bx^n)^{p+1}dx. \quad (1)$$

To bring this result into the required form, we place

$$x^{m-n}(a+bx^n)^{p+1} = x^{m-n}(a+bx^n)(a+bx^n)^p$$

whence

$$= (ax^{m-n}+bx^m)(a+bx^n)^p,$$

$$\int x^{m-n}(a+bx^n)^{p+1}dx = a\int x^{m-n}(a+bx^n)^p\,dx + b\int x^m(a+bx^n)^p\,dx.$$

Substituting this value in (1), we have

$$\int x^m(a+bx^n)^p\,dx = \frac{x^{m-n+1}(a+bx^n)^{p+1}}{nb(p+1)} - \frac{m-n+1}{nb(p+1)}\left\{a\int x^{m-n}(a+bx^n)^p\,dx + b\int x^m(a+bx^n)^p\,dx\right\}. \quad (2)$$

Solving (2) for $\int x^m(a+bx^n)^p\,dx$, we have formula (1), § 71.

If we solve the equation defining formula (1), § 71, for

$$\int x^{m-n}(a+bx^n)^p\,dx,$$

and then replace $m-n$ by m, the result is formula (3), § 71.

To derive (2), § 71, we integrate by parts, letting $(a + bx^n)^p = u$, and $x^m dx = dv$. Then

$$du = pnbx^{n-1}(a + bx^n)^{p-1}dx, \quad \text{and} \quad v = \frac{x^{m+1}}{m+1}.$$

Then

$$\int x^m (a + bx^n)^p dx$$
$$= \frac{x^{m+1}(a + bx^n)^p}{m+1} - \frac{npb}{m+1}\int x^{m+n}(a + bx^n)^{p-1}dx. \quad (3)$$

To bring this result into the required form we place

$$x^{m+n}(a + bx^n)^{p-1}$$
$$= x^m \left(\frac{a + bx^n - a}{b}\right)(a + bx^n)^{p-1}$$
$$= \frac{1}{b}\{x^m(a + bx^n)^p - ax^m(a + bx^n)^{p-1}\},$$

whence

$$\int x^{m+n}(a + bx^n)^{p-1}dx$$
$$= \frac{1}{b}\int x^m(a + bx^n)^p dx - \frac{a}{b}\int x^m(a + bx^n)^{p-1}dx.$$

Substituting this value in (3), we have

$$\int x^m(a + bx^n)^p dx$$
$$= \frac{x^{m+1}(a + bx^n)^p}{m+1} - \frac{np}{m+1}\left\{\int x^m(a + bx^n)^p dx\right.$$
$$\left. - a\int x^m(a + bx^n)^{p-1}dx\right\}. \quad (4)$$

Solving (4) for $\int x^m(a + bx^n)^p dx$, we have formula (2), § 71.

If we solve the equation defining (2), § 71, for $\int x^m(a + bx^n)^{p-1}dx$, and replace $p - 1$ by p, the result is formula (4), § 71.

73. We will now apply these reduction formulas to the evaluation of a few integrals. Many of these can also be evaluated by substitution, without the use of the reduction formulas.

Ex. 1. Find the value of $\int x^3\sqrt{a^2+x^2}\,dx$.

Applying formula (1), § 71, we make the given integral depend upon $\int x\sqrt{a^2+x^2}\,dx$, an integral which can be evaluated by the elementary formulas. The work is:

$$\begin{aligned}\int x^3\sqrt{a^2+x^2}\,dx &= \tfrac{1}{5}x^2(a^2+x^2)^{\frac{3}{2}} - \tfrac{2}{5}a^2\int x(a^2+x^2)^{\frac{1}{2}}\,dx\\ &= \tfrac{1}{5}x^2(a^2+x^2)^{\frac{3}{2}} - \tfrac{2}{15}a^2(a^2+x^2)^{\frac{3}{2}} + C\\ &= \tfrac{1}{15}(3x^2-2a^2)(a^2+x^2)^{\frac{3}{2}} + C.\end{aligned}$$

Ex. 2. Find the value of $\int\frac{dx}{x^2\sqrt{a^2-x^2}}$.

Applying (3), § 71, we have

$$\int\frac{dx}{x^2\sqrt{a^2-x^2}} = \frac{x^{-1}(a^2-x^2)^{\frac{1}{2}}}{-a^2} - 0\int\frac{dx}{\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x} + C.$$

Ex. 3. Find the value of $\int\frac{(a^2-x^2)^{\frac{3}{2}}}{x^2}\,dx$

Applying (3), § 71, we have

$$\int\frac{(a^2-x^2)^{\frac{3}{2}}}{x^2}\,dx = \frac{x^{-1}(a^2-x^2)^{\frac{5}{2}}}{-a^2} - \frac{4}{a^2}\int(a^2-x^2)^{\frac{3}{2}}\,dx. \tag{1}$$

Applying (2), § 71, to the integral in the right-hand member of (1), we have

$$\int(a^2-x^2)^{\frac{3}{2}}\,dx = \tfrac{1}{4}x(a^2-x^2)^{\frac{3}{2}} + \tfrac{3}{4}a^2\int(a^2-x^2)^{\frac{1}{2}}\,dx. \tag{2}$$

Applying (2), § 71, again, we have

$$\int(a^2-x^2)^{\frac{1}{2}}\,dx = \frac{1}{2}x(a^2-x^2)^{\frac{1}{2}} + \frac{1}{2}a^2\int\frac{dx}{\sqrt{a^2-x^2}}, \tag{3}$$

and
$$\int\frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C.$$

Substituting back, we have finally

$$\begin{aligned}\int\frac{(a^2-x^2)^{\frac{3}{2}}}{x^2}\,dx = -\frac{(a^2-x^2)^{\frac{5}{2}}}{a^2x} - \frac{x(a^2-x^2)^{\frac{3}{2}}}{a^2} - \frac{3}{2}x(a^2-x^2)^{\frac{1}{2}}\\ - \frac{3}{2}a^2\sin^{-1}\frac{x}{a} + C.\end{aligned}$$

Ex. 4. Find the value of $\int\frac{dx}{(x^2+a^2)^n}$, where n is a positive integer.

Applying (4), § 71, we have

$$\begin{aligned}\int\frac{dx}{(x^2+a^2)^n} &= -\frac{x(x^2+a^2)^{-n+1}}{2(-n+1)a^2} + \frac{-2n+3}{2(-n+1)a^2}\int\frac{dx}{(x^2+a^2)^{n-1}}\\ &= \frac{1}{2(n-1)a^2}\left[\frac{x}{(x^2+a^2)^{n-1}} + (2n-3)\int\frac{dx}{(x^2+a^2)^{n-1}}\right].\end{aligned}$$

This formula is the one which is sometimes used in integrating rational fractions.

Ex. 5. Find the value of $\int \frac{x\,dx}{\sqrt{2\,ax - x^2}}$.

If we divide both numerator and denominator of the integrand by $x^{\frac{1}{2}}$, it becomes $\frac{x^{\frac{1}{2}}}{\sqrt{2\,a - x}}$, a form to which we can apply a reduction formula. Applying (1), § 71, we have

$$\int \frac{x\,dx}{\sqrt{2\,ax - x^2}} = \int x^{\frac{1}{2}}(2\,a - x)^{-\frac{1}{2}}\,dx$$

$$= -\,x^{\frac{1}{2}}(2\,a - x)^{\frac{1}{2}} + a\int x^{-\frac{1}{2}}(2\,a - x)^{-\frac{1}{2}}\,dx.$$

But $$\int x^{-\frac{1}{2}}(2\,a - x)^{-\frac{1}{2}}\,dx = \int \frac{dx}{\sqrt{2\,ax - x^2}}$$

$$= \int \frac{dx}{\sqrt{a^2 - (x - a)^2}} = \sin^{-1}\frac{x - a}{a} + C.$$

Therefore $$\int \frac{x\,dx}{\sqrt{2\,ax - x^2}} = -\sqrt{2\,ax - x^2} + a\sin^{-1}\frac{x - a}{a} + C.$$

Ex. 6. Find the value of $\int \frac{dx}{(2\,ax - x^2)^{\frac{3}{2}}}$.

Writing $2\,ax - x^2$ as $a^2 - (x - a)^2$, and noting that $dx = d(x - a)$, we can apply (4), § 71, to advantage. The result is

$$\int \frac{dx}{(2\,ax - x^2)^{\frac{3}{2}}} = \int [a^2 - (x - a)^2]^{-\frac{3}{2}}\,dx$$

$$= -\frac{(x - a)\,[a^2 - (x - a)^2]^{-\frac{1}{2}}}{-\,a^2} + 0\int [a^2 - (x - a)^2]^{-\frac{1}{2}}\,dx$$

$$= \frac{x - a}{a^2\sqrt{2\,ax - x^2}} + C.$$

74. Trigonometric reduction formulas. It was proved in § 67 that

$$\int \tan^n x\,dx = \frac{1}{n - 1}\tan^{n-1} x - \int \tan^{n-2} x\,dx. \qquad (1)$$

Similarly, $$\int \operatorname{ctn}^n x\,dx = -\frac{1}{n - 1}\operatorname{ctn}^{n-1} x - \int \operatorname{ctn}^{n-2} x\,dx. \qquad (2)$$

Formulas (1) and (2) are evidently reduction formulas for the integration of this particular type of integrand. There are four others which we shall derive, i.e.

$$\int \sin^m x \cos^n x\, dx$$
$$= \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n}\int \sin^m x \cos^{n-2} x\, dx, \qquad (3)$$

$$\int \sin^m x \cos^n x\, dx$$
$$= -\frac{\sin^{m+1} x \cos^{n+1} x}{n+1} + \frac{m+n+2}{n+1}\int \sin^m x \cos^{n+2} x\, dx, \quad (4)$$

$$\int \sin^m x \cos^n x\, dx$$
$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} x \cos^n x\, dx, \qquad (5)$$

$$\int \sin^m x \cos^n x\, dx$$
$$= \frac{\sin^{m+1} x \cos^{n+1} x}{m+1} + \frac{m+n+2}{m+1}\int \sin^{m+2} x \cos^n x\, dx. \qquad (6)$$

These formulas are useful when m and n are integers, either positive or negative, or zero.

Formulas (3) and (5) fail if $m + n = 0$; but in that case the integral can be placed under (1) or (2), by expressing the integrand in terms of $\tan x$ or $\operatorname{ctn} x$. Formula (4) fails when $n + 1 = 0$ and formula (6) fails when $m + 1 = 0$; but in these cases the integration can be performed by the methods of § 66.

To derive (3) we place $\sin^m x \cos^n x\, dx = \cos^{n-1} x (\sin^m x \cos x\, dx)$ and let $\cos^{n-1} x = u$ and $\sin^m x \cos x\, dx = dv$ for integration by parts. As a result

$$\int \sin^m x \cos^n x\, dx$$
$$= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1}\int \sin^{m+2} x \cos^{n-2} x\, dx. \qquad (1)$$

To bring this into the required form we place

$$\sin^{m+2} x \cos^{n-2} x$$
$$= \sin^m x (1 - \cos^2 x) \cos^{n-2} x = \sin^m x \cos^{n-2} x - \sin^m x \cos^n x.$$

Then

$$\int \sin^{m+2} x \cos^{n-2} x\, dx$$
$$= \int \sin^m x \cos^{n-2} x\, dx - \int \sin^m x \cos^n x\, dx.$$

Substituting this value in (1), we have

$$\begin{aligned}\int \sin^m x \cos^n x\, dx &= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1}\int \sin^m x \cos^{n-2} x\, dx \\ &\quad - \frac{n-1}{m+1}\int \sin^m x \cos^n x\, dx. \qquad (2)\end{aligned}$$

Solving (2) for $\int \sin^m x \cos^n x\, dx$, we obtain formula (3).

If we solve formula (3) for $\int \sin^m x \cos^{n-2} x\, dx$ and then replace n by $n+2$, the result is formula (4).

The derivations of formulas (5) and (6) are left to the student.

Ex. 1. Find the value of $\int \sin^3 x \cos^2 x\, dx$.

By formula (5),

$$\int \sin^3 x \cos^2 x\, dx = -\tfrac{1}{5}\sin^2 x \cos^3 x + \tfrac{2}{5}\int \sin x \cos^2 x\, dx,$$

and
$$\int \sin x \cos^2 x\, dx = -\tfrac{1}{3}\cos^3 x,$$

by the elementary integrals.

Therefore $\int \sin^3 x \cos^2 x\, dx = -\tfrac{1}{15}\cos^3 x\,(3\sin^2 x + 2) + C.$

Ex. 2. Find the value of $\int \cos^4 x\, dx$.

Applying formula (3), and noting that $m = 0$, we have

$$\int \cos^4 x\, dx = \tfrac{1}{4}\sin x \cos^3 x + \tfrac{3}{4}\int \cos^2 x\, dx.$$

Applying formula (3) again, we have

$$\int \cos^2 x\, dx = \tfrac{1}{2}\sin x \cos x + \tfrac{1}{2}\int dx = \tfrac{1}{2}\sin x \cos x + \tfrac{1}{2}x.$$

Therefore, by substitution,

$$\int \cos^4 x\, dx = \tfrac{1}{4}\sin x \cos^3 x + \tfrac{3}{8}\sin x \cos x + \tfrac{3}{8}x + C.$$

75. Use of tables of integrals. In Chap. II we have evaluated many simple integrals by bringing them under the fundamental formulas collected in § 17, and in Chaps. VI and VII methods of dealing with more complex integrals have been discussed. But in the solution of problems involving integration it is found that some integrals occur frequently. If these integrals are tabulated with the fundamental integrals, it is evident that the work of integration may be considerably lightened by reference to such a table. Accordingly the reader is advised to acquire facility in the use of a table of integrals.

No table of integrals will be inserted here, but the reader is referred to Professor B. O. Peirce's "Short Table of Integrals."

PROBLEMS

Find the values of the following integrals:

1. $\int \frac{\sqrt[3]{x}\,dx}{\sqrt[3]{x^2}+1}$.

2. $\int \frac{\sqrt{x}\,dx}{x-1}$.

3. $\int \frac{\sqrt{x}+1}{\sqrt[4]{x^3}+\sqrt{x}}\,dx$.

4. $\int \frac{(x-2)^{\frac{1}{2}}-(x-2)^{\frac{1}{3}}}{(x-2)^{\frac{1}{6}}-(x-2)^{\frac{1}{3}}}\,dx$.

5. $\int \frac{x\,dx}{(x+1)^{\frac{1}{3}}-(x+1)^{\frac{2}{3}}}$.

6. $\int \frac{\sqrt{1-x}}{1+x}\,dx$.

7. $\int \frac{\sqrt[3]{1+3x}-2}{1+3x+2\sqrt[6]{(1+3x)^5}}\,dx$.

8. $\int \frac{x+\sqrt{1+x}}{x-\sqrt{1+x}}\,dx$.

9. $\int \frac{\sqrt{x}\,dx}{1+\sqrt{x}}$.

10. $\int \frac{(1+x^6)^{\frac{1}{3}}}{x^9}\,dx$.

11. $\int \frac{dx}{(1+2x^5)^{\frac{6}{5}}}$.

12. $\int \frac{dx}{x^2(3+x^3)^{\frac{2}{3}}}$.

13. $\int \frac{x^5\,dx}{(x^3+3)^{\frac{4}{3}}}$.

14. $\int x^9(3x^5+1)^{\frac{5}{3}}\,dx$.

15. $\int x^7(2+3x^4)^{\frac{2}{3}}\,dx$.

16. $\int \frac{dx}{x(3x^3+2)^{\frac{3}{2}}}$.

17. $\int x(4+x^4)^{\frac{1}{2}}\,dx$.

18. $\int \frac{dx}{x\sqrt{x^2+3x-2}}$.

19. $\int \frac{dx}{x\sqrt{2x^2+3x+5}}$.

20. $\int \frac{dx}{x\sqrt{3+2x-x^2}}$.

21. $\int \frac{dx}{(x^2+x+1)^{\frac{3}{2}}}$.

22. $\int \frac{x\,dx}{\sqrt{x^2+2x+3}}$.

23. $\int \frac{dx}{x^2\sqrt{x-3x^2}}$.

24. $\int \frac{x^2\,dx}{(1-x-2x^2)^{\frac{5}{2}}}$.

25. $\int \frac{x^2\,dx}{\sqrt{(2-3x-2x^2)^3}}$.

26. $\int \sin^4 x\cos^3 x\,dx$.

27. $\int \cos^{\frac{2}{3}} 3x\sin^3 3x\,dx$.

28. $\int \sin^7 x \, dx.$

29. $\int \cos^5 x \, dx.$

30. $\int \sin^3 (2x + 1) \, dx.$

31. $\int (\sin 2x + \cos 2x)^2 dx.$

32. $\int \left(\sin \frac{x}{3} - \cos \frac{x}{3}\right)^3 dx.$

33. $\int \sin^3 ax \cos^3 ax \, dx.$

34. $\int \sin^4 x \cos^2 x \, dx.$

35. $\int \sin^4 x \, dx.$

36. $\int \cos^6 2x \, dx.$

37. $\int \sin^4 \frac{x}{2} \cos^4 \frac{x}{2} \, dx.$

38. $\int \frac{\sin^3 2x}{\sqrt[3]{\cos^2 2x}} dx.$

39. $\int \frac{\sin^4 \frac{x}{2}}{\cos \frac{x}{2}} dx.$

40. $\int \frac{\cos^2 4x}{\sin^3 4x} dx.$

41. $\int \frac{dx}{\sin^3 2x}.$

42. $\int \frac{dx}{\cos^5 3x}.$

43. $\int \tan^3 3x \, dx.$

44. $\int \text{ctn}^5 3x \, dx.$

45. $\int \tan^4 \frac{x}{2} \, dx.$

46. $\int \text{ctn}^6 \frac{x}{3} \, dx.$

47. $\int (\text{ctn}\, x + \tan x)^3 dx.$

48. $\int \sec^6 \frac{x}{2} \, dx.$

49. $\int \csc^4 \frac{x}{4} \, dx.$

50. $\int \csc^3 3x \, dx.$

51. $\int \sec^3 2x \, dx.$

52. $\int \csc^5 \frac{x}{5} \, dx.$

53. $\int \text{ctn}^3 \frac{x}{3} \csc^4 \frac{x}{3} \, dx.$

54. $\int \frac{\text{ctn}^5 ax}{\csc ax} dx.$

55. $\int \tan \frac{x}{2} \sec^5 \frac{x}{2} \, dx.$

56. $\int \tan^5 \frac{x}{5} \sqrt[5]{\sec \frac{x}{5}} \, dx.$

57. $\int \tan^{\frac{5}{2}} x \sec^6 x \, dx.$

58. $\int \sec^3 5x \tan^2 5x \, dx.$

59. $\int \frac{dx}{5 + 4 \cos x}.$

60. $\int \frac{dx}{3 \cos x + 5}.$

61. $\int \frac{dx}{3 + \sin x}.$

62. $\int \frac{dx}{3 \sin x - 2 \cos x}.$

63. $\int \frac{dx}{3 - 4 \sin 2x}.$

64. $\int (x^2 + a^2)^{\frac{3}{2}} dx.$

65. $\int (a^2 - x^2)^{\frac{3}{2}} dx.$

66. $\int \frac{x^2 \, dx}{\sqrt{x^2 + a^2}}.$

67. $\int \frac{dx}{x^3 \sqrt{x^2 + a^2}}.$

68. $\int \frac{dx}{x^3 \sqrt{a^2 - x^2}}.$

69. $\int x^2 \sqrt{x^2 + a^2} \, dx.$

70. $\int x^2 \sqrt{a^2 - x^2} \, dx.$

71. $\int x^3 \sqrt{a^2 - x^2} \, dx.$

72. $\int \frac{x^2 \, dx}{(a^2 + x^2)^2}.$

73. $\int \frac{dx}{(1 + x^3)^{\frac{4}{3}}}.$

74. $\int \frac{x^5 \, dx}{(1 + x^3)^{\frac{1}{2}}}.$

75. $\int \frac{dx}{x^2 (1 + x^4)^{\frac{3}{4}}}.$

76. $\int \frac{dx}{x^2 (1 + x^3)^{\frac{2}{3}}}.$

77. $\int \frac{dx}{x \sqrt{2ax - x^2}}.$

78. $\int \frac{\sqrt{2ax - x^2}}{x} dx.$

79. $\int x \sqrt{2ax - x^2} \, dx.$

80. $\int \frac{x^2 \, dx}{\sqrt{2ax - x^2}}.$

81. $\int \sin^2 x \cos^4 x \, dx.$

82. $\int \frac{dx}{\sin^2 x \cos^3 x}.$

83. $\int \frac{dx}{\sin^3 x}.$

84. $\int \frac{dx}{\cos^4 3x}.$

85. $\int \frac{\sin^3 x}{\cos^5 x} dx.$

86. $\int \frac{\cos^4 2x}{\sin^3 2x} dx.$

87. Find the area bounded by the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and the chord $x = h$.

88. Find the area of the loop of the curve $cy^2 = (x - a)(x - b)^2$, $(a < b)$.

89. Find the total area of the curve $a^2y^2 = x^3(2a - x)$.

90. Find the area of the loop of the curve $16a^4y^2 = b^2x^2(a^2 - 2ax)$.

91. Find the area of a loop of the curve $y^2(a^2 + x^2) = x^2(a^2 - x^2)$.

92. Find the area of the loop of the curve $(x + y)^2 = y^2(y + 1)$.

93. Find the area inclosed by the four-cusped hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

94. Find the area inclosed by the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$.

95. Find the area included between the cissoid $y^2 = \frac{x^3}{2a - x}$ and its asymptote.

96. Find the area of the loop of the strophoid $y^2 = \frac{x^2(a + x)}{a - x}$.

97. Find the area bounded by the strophoid $y^2 = \frac{x^2(a + x)}{a - x}$ and its asymptote, excluding the area of the loop.

98. Find the area of a loop of the curve $r = a \cos n\theta + b \sin n\theta$.

99. Find the entire area bounded by the curve $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$.

100. Find the area of the loop of the Folium of Descartes, $x^3 + y^3 - 3axy = 0$, by the use of polar coördinates.

101. Find the length of the spiral of Archimedes, $r = a\theta$, from the pole to the end of the first revolution.

102. Find the length of the curve $8a^3y = x^4 + 6a^2x^2$ from the origin to the point $x = 2a$.

103. Find the volume of the solid formed by revolving about OX the figure bounded by OX and an arch of the cycloid $x = a(\phi - \sin\phi)$, $y = a(1 - \cos\phi)$.

104. Find the volume of the solid bounded by the surface formed by revolving the witch $y = \frac{8a^3}{x^2 + 4a^2}$ about its asymptote.

105. Find the volume of the solid generated by revolving about the asymptote of the cissoid $y^2 = \frac{x^3}{2a - x}$ the plane area bounded by the curve and the asymptote.

106. A right circular cylinder of radius a is intersected by two planes, the first of which is perpendicular to the axis of the cylinder, and the second of which makes an angle θ with the first. Find the volume of the portion of the cylinder included between these two planes, if their line of intersection is tangent to the circle cut from the cylinder by the first plane.

107. An ellipse and a parabola lie in two parallel horizontal planes, the distance between which is h, and are situated so that a vertex of the ellipse is vertically over the vertex of the parabola, the major axis of the ellipse being parallel to and in the same direction as the axis of the parabola. A trapezoid, having for its upper base a double ordinate of the ellipse and for its lower base a double ordinate of the parabola, generates a solid, whose upper base is the ellipse, by moving with its plane always perpendicular to the two parallel planes. Find the volume of the solid, the semiaxes of the ellipse being a and b, and the distance from the vertex to the focus of the parabola being $\frac{a}{4}$.

108. Find the center of gravity of the arc of the cycloid $x = a(\phi - \sin\phi)$, $y = a(1 - \cos\phi)$, between the first two cusps.

109. Find the center of gravity of the plane surface bounded by the first arch of the cycloid and the axis of x.

110. Find the center of gravity of the plane surface bounded by the two circles, $x^2 + y^2 = a^2$ and $x^2 + y^2 - 2ax = 0$, and the axis of x.

111. Find the center of gravity of that part of the plane surface bounded by the four-cusped hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, which is in the first quadrant.

112. Find the center of gravity of the surface generated by the revolution about the initial line of one of the loops of the lemniscate $r^2 = 2a^2\cos 2\theta$.

CHAPTER VIII

INTEGRATION OF SIMPLE DIFFERENTIAL EQUATIONS

76. Definitions. A differential equation is an equation which contains derivatives. Such an equation can be changed into one which contains differentials, and hence its name, but this change is usually not desirable unless the equation contains the first derivative only.

A differential equation containing x, y, and derivatives of y with respect to x, is said to be *solved* or *integrated* when a relation between x and y, but not containing the derivatives, has been found, which, if substituted in the differential equation, reduces it to an identity.

The manner in which differential equations can occur in practice and methods for their integration are illustrated in the two following examples:

Ex. 1. Required a curve such that the length of the tangent from any point to its intersection with OY is constant.

Let $P(x, y)$ (fig. 41) be any point on the required curve. Then the equation of the tangent at P is

$$Y - y = \frac{dy}{dx}(X - x),$$

where (X, Y) are the variable coördinates of a moving point of the tangent, (x, y) the constant coördinates of a fixed point on the tangent (the point of tangency), and $\frac{dy}{dx}$ is derived from the, as yet unknown, equation of the curve. The coördinates of R, where the tangent intersects OY, are then $X = 0$, $Y = y - \frac{dy}{dx}x$, and the length of PR is $\sqrt{x^2 + x^2\left(\frac{dy}{dx}\right)^2}$.

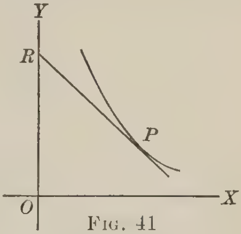

FIG. 41

Representing by a the constant length of the tangent, we have

$$x^2 + x^2\left(\frac{dy}{dx}\right)^2 = a^2,$$

or

$$\frac{dy}{dx} = \pm\frac{\sqrt{a^2 - x^2}}{x}, \tag{1}$$

which is the differential equation of the required curve. Its solution is clearly

$$y = \pm \int \frac{\sqrt{a^2 - x^2}}{x} dx + C$$
$$= \pm \sqrt{a^2 - x^2} + \frac{a}{2} \log \frac{a \mp \sqrt{a^2 - x^2}}{a \pm \sqrt{a^2 - x^2}} + C. \quad (2)$$

The arbitrary constant C shows that there are an infinite number of curves which satisfy the conditions of the problem. Assuming a fixed value for C, we see from (1) and (2) that the curve is symmetrical with respect to OY, that x^2 cannot be greater than a^2, that $\frac{dy}{dx} = 0$ and $y = C$ when $x = a$, and that $\frac{dy}{dx}$ becomes infinite as x approaches zero.

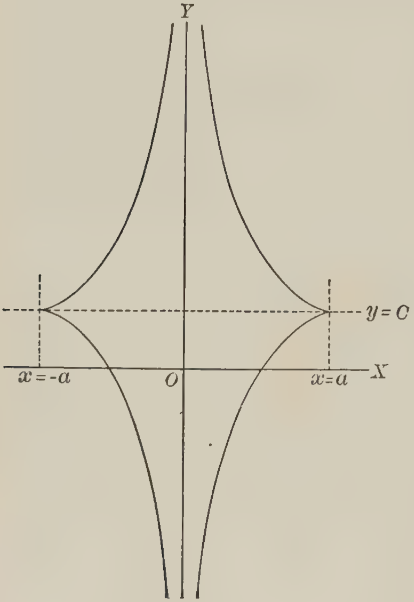

Fig. 42

From these facts and the defining property the curve is easily sketched, as shown in fig. 42. The curve is called the *tractrix* (I, p. 299).

Ex. 2. A uniform cable is suspended from two fixed points. Required the curve in which it hangs.

Let A (fig. 43) be the lowest point, and P any point on the required curve, and let PT be the tangent at P. Since the cable is in equilibrium, we may consider the portion AP as a rigid body acted on by three forces, — the tension t at P acting along PT, the tension h at A acting horizontally, and the weight of AP acting vertically. Since the cable is uniform, the weight of AP is ρs, where s is the length of AP and ρ the weight of the cable per unit of length. Equating the horizontal components of these forces, we have

$$t \cos \phi = h,$$

and equating the vertical components, we have

$$t \sin \phi = \rho s.$$

From these two equations we have

$$\tan \phi = \frac{\rho}{h} s,$$

or

$$a \frac{dy}{dx} = s,$$

where $\frac{h}{\rho} = a$, a constant.

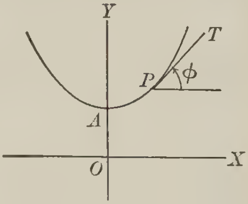

Fig. 43

This equation contains three variables, x, y, and s, but by differentiating with respect to x we have (I, § 105, (4))

$$a\frac{d^2y}{dx^2}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}, \tag{1}$$

the differential equation of the required path.

To solve (1), place $\frac{dy}{dx}=p$. Then (1) becomes

$$a\frac{dp}{dx}=\sqrt{1+p^2},$$

or

$$\frac{dp}{\sqrt{1+p^2}}=\frac{dx}{a},$$

whence

$$\log\left(p+\sqrt{1+p^2}\right)=\frac{x}{a}+C. \tag{2}$$

Since A is the lowest point of the curve, we know that when $x=0$, $p=0$. Hence, in (2), $C=0$, and we have

$$p+\sqrt{1+p^2}=e^{\frac{x}{a}},$$

or

$$p=\tfrac{1}{2}\left(e^{\frac{x}{a}}-e^{-\frac{x}{a}}\right),$$

whence, since $p=\frac{dy}{dx}$,

$$y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)+C'.$$

The value of C' depends upon the position of OX, since $y=a+C'$ when $x=0$. We can, if we wish, so take OX that $OA=a$. Then $C'=0$, and we have, finally,

$$y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right),$$

the equation of the *catenary* (I, p. 281).

The *order* of a differential equation is equal to that of the derivative of the highest order in it. Hence (1), Ex. 1, is of the first order, and (1), Ex. 2, is of the second order.

The simplest differential equation is that of the first order and of the first degree in the derivative, the general form of which is

$$M+N\frac{dy}{dx}=0,$$

or

$$M\,dx+N\,dy=0, \tag{1}$$

where M and N are functions of x and y, or constants.

We shall consider three cases in which this equation is readily solved. They are:

1. When the variables can be easily separated.
2. When M and N are homogeneous functions of x and y of the same degree.
3. When the equation is linear.

77. The equation $Mdx + Ndy = 0$ when the variables can be separated. If the equation (1), § 76, is in the form

$$f_1(x)\,dx + f_2(y)\,dy = 0,$$

it is said that the variables are separated. The solution is then evidently

$$\int f_1(x)\,dx + \int f_2(y)\,dy = c,$$

where c is an arbitrary constant.

The variables can be separated if M and N can each be factored into two factors, one of which is a function of x alone, and the other a function of y alone. The equation may then be divided by the factor of M which contains y multiplied by the factor of N which contains x.

Ex. 1. $dy = f(x)\,dx$.

From this follows $y = \int f(x)\,dx + c$.

Any indefinite integral may be regarded as the solution of a differential equation with separated variables.

Ex. 2. $\sqrt{1-y^2}\,dx + \sqrt{1-x^2}\,dy = 0$.

This equation may be written

$$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} = 0,$$

whence, by integration, $$\sin^{-1}x + \sin^{-1}y = c. \tag{1}$$

This solution can be put into another form, thus: Let $\sin^{-1}x = \phi$ and $\sin^{-1}y = \psi$. Equation (1) is then $\phi + \psi = c$, whence $\sin(\phi + \psi) = \sin c$; that is, $\sin\phi\cos\psi + \cos\phi\sin\psi = k$, where k is a constant. But $\sin\phi = x$, $\sin\psi = y$, $\cos\phi = \sqrt{1-x^2}$, $\cos\psi = \sqrt{1-y^2}$; hence we have

$$x\sqrt{1-y^2} + y\sqrt{1-x^2} = k. \tag{2}$$

In (1) and (2) we have not two solutions, but two forms of the same solution, of the differential equation. It is, in fact, an important theorem that the differential equation $Mdx + Ndy = 0$ has only one solution involving an arbitrary constant. The student must be prepared, however, to meet different forms of the same solution.

Ex. 3. $(1-x^2)\dfrac{dy}{dx}+xy=ax.$

This is readily written as

$$(1-x^2)\,dy+x(y-a)\,dx=0,$$

or

$$\frac{dy}{y-a}+\frac{x\,dx}{1-x^2}=0,$$

whence, by integration,

$$\log(y-a)-\tfrac{1}{2}\log(1-x^2)=c,$$

which is the same as

$$\log\frac{y-a}{\sqrt{1-x^2}}=c,$$

and this may be written

$$y-a=k\sqrt{1-x^2}.$$

78. The homogeneous equation $Mdx+Ndy=0$. A polynomial in x and y is said to be homogeneous when the sum of the exponents of those letters in each term is the same. Thus $ax^2+bxy+cy^2$ is homogeneous of the second degree, $ax^3+bx^2y+cxy^2+ey^3$ is homogeneous of the third degree. If, in such a polynomial, we place $y=vx$, it becomes $x^nf(v)$ where n is the degree of the polynomial. Thus

$$ax^2+bxy+cy^2=x^2(a+bv+cv^2),$$

$$ax^3+bx^2y+cxy^2+ey^3=x^3(a+bv+cv^2+ev^3).$$

This property enables us to extend the idea of homogeneity to functions which are not polynomials. Representing by $f(x,y)$ a function of x and y, we shall say that $f(x,y)$ is a homogeneous function of x and y of the nth degree, if, when we place $y=vx$, $f(x,y)=x^nF(v)$. Thus $\sqrt{x^2+y^2}$ is homogeneous of the first degree, since $\sqrt{x^2+y^2}=x\sqrt{1+v^2}$, and $\log\dfrac{y}{x}$ is homogeneous of degree 0, since $\log\dfrac{y}{x}=\log v=x^0\log v.$

When M and N are homogeneous functions of the same degree, the equation

$$Mdx+Ndy=0$$

is said to be homogeneous and can be solved as follows:

Place $y=vx$. Then $dy=vdx+xdv$ and the differential equation becomes

$$x^nf_1(v)\,dx+x^nf_2(v)\,(vdx+xdv)=0,$$

or

$$[f_1(v)+vf_2(v)]\,dx+xf_2(v)\,dv=0. \tag{1}$$

If $f_1(v)+vf_2(v)\neq 0$, this can be written

$$\frac{dx}{x}+\frac{f_2(v)\,dv}{f_1(v)+vf_2(v)}=0,$$

where the variables are now separated and the equation may be solved as in § 77.

If $f_1(v)+vf_2(v)=0$, (1) becomes $dv=0$, whence $v=c$ and $y=cx$.

Ex. $(x^2-y^2)\,dx+2\,xy\,dy=0$.

Place $y=vx$. There results

$$(1-v^2)\,dx+2\,v(x\,dv+v\,dx)=0,$$

or

$$\frac{dx}{x}+\frac{2\,v\,dv}{1+v^2}=0.$$

Integrating, we have $\log x+\log(1+v^2)=c'$,

whence $x(1+v^2)=c$,

or $x^2+y^2=cx$.

79. The equation

$$(a_1x+b_1y+c_1)\,dx+(a_2x+b_2y+c_2)\,dy=0 \tag{1}$$

is not homogeneous, but it can usually be made so, as follows:

Place $x=x'+h,\quad y=y'+k.$ (2)

Equation (1) becomes

$$\begin{aligned}(a_1x'+b_1y'+a_1h+b_1k+c_1)\,dx'\\ +(a_2x'+b_2y'+a_2h+b_2k+c_2)\,dy'=0.\end{aligned} \tag{3}$$

If, now, we can determine h and k so that

$$\left.\begin{aligned}a_1h+b_1k+c_1=0\\ a_2h+b_2k+c_2=0\end{aligned}\right\}, \tag{4}$$

(3) becomes $(a_1x'+b_1y')\,dx'+(a_2x'+b_2y')\,dy'=0$,

which is homogeneous and can be solved as in § 78.

Now (4) cannot be solved if $a_1b_2-a_2b_1=0$. In this case, $\frac{a_2}{a_1}=\frac{b_2}{b_1}=k$, where k is some constant. Equation (1) is then of the form

$$(a_1x+b_1y+c_1)\,dx+[k(a_1x+b_1y)+c_2]\,dy=0, \tag{5}$$

so that, if we place $a_1x+b_1y=x'$, (5) becomes

$$(x'+c_1)\,dx+(kx'+c_2)\frac{dx'-a_1dx}{b_1}=0,$$

which is

$$dx+\frac{kx'+c_2}{(b_1-a_1k)x'+b_1c_1-a_1c_2}dx'=0,$$

and the variables are separated.

Hence (1) can always be solved.

80. The linear equation of the first order. The equation

$$\frac{dy}{dx}+f_1(x)y=f_2(x), \tag{1}$$

where $f_1(x)$ and $f_2(x)$ may reduce to constants but cannot contain y, is called a linear equation of the first order. It is a special case of $Mdx+Ndy=0$, where $M=f_1(x)y-f_2(x)$, $N=1$.

To solve the equation we will try the experiment of placing

$$y = uv,$$

where u and v are unknown functions of x to be determined later in any way which may be advantageous. Then (1) becomes

$$u\frac{dv}{dx} + v\frac{du}{dx} + f_1(x)uv = f_2(x),$$

or

$$v\left[\frac{du}{dx} + f_1(x)u\right] + u\frac{dv}{dx} = f_2(x). \qquad (2)$$

Let us now determine u so that the coefficient of v in (2) shall be zero. We have

$$\frac{du}{dx} + f_1(x)u = 0,$$

or

$$\frac{du}{u} + f_1(x)dx = 0,$$

of which the general solution is

$$\log u + \int f_1(x)dx = c.$$

Since, however, all we need is a particular function which will make the coefficient of v in (2) equal to zero, we may take $c = 0$.

Then

$$\log u = -\int f_1(x)dx,$$

or

$$u = e^{-\int f_1(x)dx}. \qquad (3)$$

With this value of u, (2) becomes

$$e^{-\int f_1(x)dx}\frac{dv}{dx} = f_2(x),$$

or

$$\frac{dv}{dx} = e^{\int f_1(x)dx}f_2(x),$$

and

$$v = \int e^{\int f_1(x)dx}f_2(x)dx + c. \qquad (4)$$

Whence, finally, since $y = uv$,

$$y = e^{-\int f_1(x)dx}\int e^{\int f_1(x)dx}f_2(x)dx + ce^{-\int f_1(x)dx}. \qquad (5)$$

Ex. $(1-x^2)\dfrac{dy}{dx}+xy=ax.$

Rewriting this equation as

$$\frac{dy}{dx}+\frac{x}{1-x^2}y=\frac{ax}{1-x^2},$$

we recognize a linear equation in which

$$f_1(x)=\frac{x}{1-x^2},\quad f_2(x)=\frac{ax}{1-x^2}.$$

Then $\int f_1(x)\,dx=-\log\sqrt{1-x^2}$, and $e^{\int f_1(x)dx}=e^{-\log\sqrt{1-x^2}}=\dfrac{1}{\sqrt{1-x^2}}.$

Hence
$$\begin{aligned}y&=\sqrt{1-x^2}\int\frac{ax}{(1-x^2)^{\frac{3}{2}}}dx+c\sqrt{1-x^2}\\&=a+c\sqrt{1-x^2}.\end{aligned}$$

This example is the same as Ex. 3, § 77, showing that the methods of solving an equation are not always mutually exclusive.

81. Bernouilli's equation.

The equation
$$\frac{dy}{dx}+f_1(x)y=f_2(x)y^n,$$

while not linear, can be made so, as follows:

Dividing by y^n, we have

$$y^{-n}\frac{dy}{dx}+f_1(x)y^{1-n}=f_2(x),$$

and placing $y^{1-n}=z$, and multiplying by $1-n$, we have

$$\frac{dz}{dx}+(1-n)f_1(x)z=(1-n)f_2(x),$$

a linear equation.

Ex. $\dfrac{dy}{dx}-\dfrac{y}{x}=x^2y^4.$

That is,
$$y^{-4}\frac{dy}{dx}-\frac{1}{x}y^{-3}=x^2,$$

or
$$\frac{dz}{dx}+\frac{3z}{x}=-3x^2,\quad \text{where}\quad z=y^{-3}.$$

We have now
$$\int f_1(x)dx=\int\frac{3\,dx}{x}=\log x^3,$$

whence
$$e^{\int f_1(x)dx}=x^3.$$

Hence
$$z=\frac{1}{x^3}\int(-3x^5)\,dx+\frac{c}{x^3},$$

and
$$\frac{1}{y^3}=-\frac{1}{2}x^3+\frac{c}{x^3}.$$

82. Certain equations of the second order. There are certain equations of the second order, occurring frequently in practice, which are readily integrated. These are of the four types:

1. $\dfrac{d^2y}{dx^2}=f(x).$

2. $\dfrac{d^2y}{dx^2}=f\left(x, \dfrac{dy}{dx}\right).$

3. $\dfrac{d^2y}{dx^2}=f\left(y, \dfrac{dy}{dx}\right).$

4. $\dfrac{d^2y}{dx^2}=f(y).$

We proceed to discuss these four types in order:

1. $\dfrac{d^2y}{dx^2}=f(x).$

By direct integration

$$\frac{dy}{dx}=\int f(x)\,dx+c_1,$$

$$y=\iint f(x)\,dx^2+c_1x+c_2.$$

This method is equally applicable to the equation $\dfrac{d^ny}{dx^n}=f(x).$

Ex. 1. Differential equations of this type appear in the theory of the bending of beams. Each of the forces which act on the beam, such as the loads and the reactions at the supports, has a moment about any cross section of the beam equal to the product of the force and the distance of its point of application from the section. The sum of these moments for all forces on one side of a given section is called the *bending moment* at the section. On the other hand, it is shown in the theory of beams that the bending moment is equal to $\dfrac{EI}{R}$, where E, the modulus of elasticity of the material of the beam, and I, the moment of inertia of the cross section about a horizontal line through its center, are constants, and R is the radius of curvature of the curve into which the beam is bent. Now by I, § 191,

$$\frac{1}{R}=\frac{\dfrac{d^2y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}},$$

where the axis of x is horizontal. But in most cases arising in practice $\dfrac{dy}{dx}$ is very small, and if we expand $\dfrac{1}{R}$ by the binomial theorem, thus:

$$\frac{1}{R}=\frac{d^2y}{dx^2}\left[1-\frac{3}{2}\left(\frac{dy}{dx}\right)^2+\cdots\right],$$

we may neglect all terms except the first without sensible error. Hence the bending moment is taken to be $EI\frac{d^2y}{dx^2}$. This expression equated to the bending moment as defined above gives the differential equation of the shape of the beam.

We will apply this to find the shape of a beam uniformly loaded and supported at its ends.

Let l be the distance between the supports, and w the load per foot-run. Take the origin of coördinates at the lowest point of the beam, which, by symmetry, is at its middle point. Take a plane section C (fig. 44) at a distance x from O and consider the forces at the right of C. These are the load on CB and the reaction of the support at B. The load on CB is $w\left(\frac{l}{2}-x\right)$, acting at the center of gravity of CB, which is at the distance of $\frac{\frac{l}{2}-x}{2}$ from C. Hence the moment of the load is $-\frac{w\left(\frac{l}{2}-x\right)^2}{2}$, which is taken negative, since the load acts downward. The support B supports half the load equal to $\frac{wl}{2}$. The moment of this reaction about C is therefore $\frac{wl}{2}\left(\frac{l}{2}-x\right)$. Hence we have

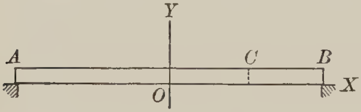

FIG. 44

$$EI\frac{d^2y}{dx^2}=\frac{wl}{2}\left(\frac{l}{2}-x\right)-\frac{w}{2}\left(\frac{l}{2}-x\right)^2=\frac{w}{2}\left(\frac{l^2}{4}-x^2\right).$$

The general solution of this equation is

$$EIy=\frac{w}{2}\left(\frac{l^2x^2}{8}-\frac{x^4}{12}\right)+c_1x+c_2.$$

But in the case of the beam, since, when $x=0$, both y and $\frac{dy}{dx}$ are 0, we have $c_1=0$, $c_2=0$.

Hence the required equation is

$$EIy=\frac{w}{2}\left(\frac{l^2x^2}{8}-\frac{x^4}{12}\right).$$

2. $\frac{d^2y}{dx^2}=f\left(x,\frac{dy}{dx}\right)$.

The essential thing here is that the equation contains $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, but does not contain y except implicitly in these derivatives. Hence if we place $\frac{dy}{dx}=p$, we have $\frac{d^2y}{dx^2}=\frac{dp}{dx}$, and the equation becomes $\frac{dp}{dx}=f(x, p)$, which is of the first order in p and x. If we can find p from this equation, we can then find y from $\frac{dy}{dx}=p$. This method has been exemplified in Ex. 2, § 76.

3. $\dfrac{d^2y}{dx^2}=f\left(y, \dfrac{dy}{dx}\right)$.

The essential thing here is that the equation contains $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$, but does not contain x. As before, we place $\dfrac{dy}{dx}=p$, but now write $\dfrac{d^2y}{dx^2}=\dfrac{dp}{dx}=\dfrac{dp}{dy}\dfrac{dy}{dx}=p\dfrac{dp}{dy}$, so that the equation becomes $p\dfrac{dp}{dy}=f(y,p)$, which is of the first order in p and y. If we can find p from this equation, we can find y from $\dfrac{dy}{dx}=p$.

Ex. 2. Find the curve for which the radius of curvature at any point is equal to the length of the portion of the normal between the point and the axis of x.

The length of the radius of curvature is $\pm\dfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}$ (I, § 192). The equation of the normal is (I, § 101)

$$Y-y=-\frac{dx}{dy}(X-x).$$

This intersects OX at the point $\left(x+y\dfrac{dy}{dx}, 0\right)$. The length of the normal is therefore $y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$.

The conditions of the problem are satisfied by either of the differential equations

$$\frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}=y\sqrt{1+\left(\frac{dy}{dx}\right)^2}, \tag{1}$$

or

$$-\frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}=y\sqrt{1+\left(\frac{dy}{dx}\right)^2}. \tag{2}$$

Placing $\dfrac{dy}{dx}=p$ and $\dfrac{d^2y}{dx^2}=p\dfrac{dp}{dy}$ in (1), we have

$$1+p^2=py\frac{dp}{dy},$$

whence

$$\frac{dy}{y}=\frac{pdp}{1+p^2}.$$

The solution of the last equation is

$$y=c_1\sqrt{1+p^2},$$

whence

$$p=\frac{\sqrt{y^2-c_1^2}}{c_1}.$$

Replacing p by $\dfrac{dy}{dx}$, we have $\dfrac{c_1 dy}{\sqrt{y^2 - c_1^2}} = dx.$

This is most neatly solved by the use of hyperbolic functions. We have (§ 15)

$$c_1 \cosh^{-1} \frac{y}{c_1} = x - c_2,$$

or

$$y = c_1 \cosh \frac{x - c_2}{c_1}$$

$$= \frac{c_1}{2}\left(e^{\frac{x-c_2}{c_1}} + e^{-\frac{x-c_2}{c_1}}\right).$$

This is the equation of a *catenary* with its vertex at the point (c_2, c_1).

If we place $\dfrac{dy}{dx} = p$, and $\dfrac{d^2y}{dx^2} = p\dfrac{dp}{dy}$ in (2), we have

$$1 + p^2 = -py\frac{dp}{dy},$$

whence

$$\frac{dy}{y} = \frac{-pdp}{1+p^2}.$$

The solution of this equation is $y = \dfrac{c_1}{\sqrt{1+p^2}},$

whence

$$p = \frac{\sqrt{c_1^2 - y^2}}{y}.$$

Replacing p by $\dfrac{dy}{dx}$, we have $\dfrac{y\,dy}{\sqrt{c_1^2 - y^2}} = dx.$

Integrating, we have $-\sqrt{c_1^2 - y^2} = x - c_2,$

or

$$(x - c_2)^2 + y^2 = c_1^2.$$

This is the equation of a *circle* with its center on OX.

4. $\dfrac{d^2y}{dx^2} = f(y).$

If we multiply both sides of this equation by $2\dfrac{dy}{dx}dx$, we have

$$2\frac{d^2y}{dx^2}\frac{dy}{dx}dx = 2f(y)\frac{dy}{dx}dx,$$

or

$$d\left[\left(\frac{dy}{dx}\right)^2\right] = 2f(y)\,dy.$$

Integrating, we have $\left(\dfrac{dy}{dx}\right)^2 = \displaystyle\int 2f(y)dy + c_1,$

whence, by separating the variables, we have

$$\int\frac{dy}{\sqrt{2\int f(y)\,dy + c_1}} = x + c_2.$$

Ex. 3. Consider the motion of a simple pendulum consisting of a particle P (fig. 45) of mass m suspended from a point C by a weightless string of length l. Let the angle $ACP = \theta$, where AC is the vertical, and let $AP = s$. By I, § 108, the force acting in the direction AP is equal to $m\frac{d^2s}{dt^2}$; but the only force acting in this direction is the component of gravity. The weight of the pendulum being mg, its component in the direction AP is equal to $-mg\sin\theta$. Hence the differential equation of the motion is

$$m\frac{d^2s}{dt^2} = -mg\sin\theta.$$

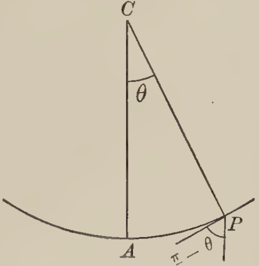

FIG. 45

We shall treat this equation first on the hypothesis that the angle through which the pendulum swings is so small that we may place $\sin\theta = \theta$, without sensible error. Then since $\theta = \frac{s}{l}$, the equation becomes

$$\frac{d^2s}{dt^2} = -\frac{g}{l}s.$$

Multiplying by $2\frac{ds}{dt}dt$, and integrating, we have

$$\left(\frac{ds}{dt}\right)^2 = c_1 - \frac{g}{l}s^2 = \frac{g}{l}(a^2 - s^2),$$

where a^2 is a new arbitrary constant. Separating the variables, we have

$$\frac{ds}{\sqrt{a^2 - s^2}} = \sqrt{\frac{g}{l}}\,dt,$$

whence
$$\sin^{-1}\frac{s}{a} = \sqrt{\frac{g}{l}}(t - t_0),$$

where t_0 is an arbitrary constant. From this, finally,

$$s = a\sin\sqrt{\frac{g}{l}}(t - t_0).$$

The physical meaning of the arbitrary constants can be given. For a is the maximum value of s; it is therefore the amplitude of the swing. When $t = t_0$, $s = 0$; hence t_0 is the time at which the pendulum passes through the vertical.

We will next integrate the equation

$$m\frac{d^2s}{dt^2} = -mg\sin\theta$$

without assuming that the arc of swing is small. Placing $s = l\theta$, multiplying by $2\frac{d\theta}{dt}dt$, and integrating, we have

$$l\left(\frac{d\theta}{dt}\right)^2 = 2g\cos\theta + c_1.$$

If, now, the pendulum does not make a complete revolution around the point of support C, $\frac{d\theta}{dt} = 0$ for some value of θ which we will call α. Hence $C = -2g\cos\alpha$, and our equation becomes

$$l\left(\frac{d\theta}{dt}\right)^2 = 2g(\cos\theta - \cos\alpha),$$

whence

$$\int_0^\theta \frac{d\theta}{\sqrt{2(\cos\theta - \cos\alpha)}} = \sqrt{\frac{g}{l}}(t - t_0),$$

where t_0 is the value of t for which $\theta = 0$. To bring the integral into a familiar form, place $\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$, $\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2}$, and let $k = \sin\frac{\alpha}{2}$. We have, then,

$$\frac{1}{2}\int_0^\theta \frac{d\theta}{\sqrt{k^2 - \sin^2\frac{\theta}{2}}} = \sqrt{\frac{g}{l}}(t - t_0).$$

Place, now, $\sin\frac{\theta}{2} = k\sin\phi$. There results

$$\int_0^\phi \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}} = \sqrt{\frac{g}{l}}(t - t_0).$$

If we measure time from the instant in which $\theta = 0$, we have

$$t = \sqrt{\frac{l}{g}}\int_0^\phi \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}.$$

The integral can be evaluated by expanding $(1 - k^2\sin^2\phi)^{-\frac{1}{2}}$ by the binomial theorem, the expansion being valid since $k^2\sin^2\phi < 1$, and a table for the value of this series is found in B. O. Peirce's Tables, p. 118.

The pendulum makes one fourth of a complete swing when θ varies from 0 to α, and ϕ from 0 to $\frac{\pi}{2}$. If $4T$ is the time of a complete swing, then

$$T = \frac{\pi}{2}\sqrt{\frac{l}{g}}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6 + \cdots\right],$$

as may be verified by the use of Ex. 2, § 27.

PROBLEMS

Solve the following equations:

1. $\frac{x\,dx}{1+y} - \frac{y\,dy}{1+x} = 0.$

2. $2x\sin y\,dx + \cos y\,dy = 0.$

3. $x\sqrt{1+y^2}\,dx + y\sqrt{1+x^2}\,dy = 0.$

4. $\sec^2 x\,dx + \tan x\tan y\,dy = 0.$

5. $(x^2y^2 - y^2)\,dx + (x^2 + y^2x^2)\,dy = 0.$

6. $(x + y)\,dx - x\,dy = 0.$

7. $(y + \sqrt{x^2 + y^2})\,dx - x\,dy = 0.$

8. $\left(\sqrt{x^2 - y^2} - y\sin^{-1}\frac{y}{x}\right)dx + x\sin^{-1}\frac{y}{x}\,dy = 0.$

9. $\sin x\sin y\,dx - \cos y\cos x\,dy = 0.$

10. $\left(x\sin\frac{y}{x}-y\cos\frac{y}{x}\right)dx+x\cos\frac{y}{x}\,dy=0.$

11. $\left(e^{\frac{y}{x}}+e^{-\frac{y}{x}}\right)(x\,dy-y\,dx)+\left(e^{\frac{y}{x}}-e^{-\frac{y}{x}}\right)x\,dx=0.$

12. $(2y^2-3xy)\,dx+(3x^2-xy+y^2)\,dy=0.$

13. $(x+2y-3)\,dx+(2x-y-1)\,dy=0.$

14. $(y+3)\,dx+(x+2y+4)\,dy=0.$

15. $(x+y)\,dx+(x+y+1)\,dy=0.$

16. $\frac{dy}{dx}+y=\sin x.$

17. $(x+1)\frac{dy}{dx}-2y=e^x(x+1)^3.$

18. $(x+1)\frac{dy}{dx}-y=(x+1)^3.$

19. $\left(x-y\cos\frac{y}{x}\right)dx+x\cos\frac{y}{x}\,dy=0.$

20. $x\frac{dy}{dx}+y=xe^x.$

21. $(y+x^2y)\,dx+(x-xy^2)\,dy=0.$

22. $x\,dy-y\,dx+\sqrt{x^2+y^2}\,dx=0.$

23. $\frac{dy}{dx}-\frac{x}{1+x^2}y=x.$

24. $\frac{dy}{dx}+\frac{y}{\sqrt{1+x^2}}=\frac{1}{\sqrt{1+x^2}}.$

25. $\frac{dy}{dx}+y\tan x=\sin 2x.$

26. $(x^2-2y^2)\,dx+(3x^2+4xy)\,dy=0.$

27. $\frac{dy}{dx}+\frac{2y}{x^2-1}=x+1.$

28. $\frac{dy}{dx}-y=\frac{x^2+1}{2y}.$

29. $x^2(1+x^2)\frac{dy}{dx}-x^3y=y^3.$

30. $(1+x^2)\frac{dy}{dx}-y+y^2=0.$

31. $xy(1+x^2)\,dy-(1-y^2)\,dx=0.$

32. $(x^2\sqrt[3]{x^3+y^3}-y^3)\,dx+xy^2\,dy=0.$

33. $\frac{dy}{dx}+\frac{e^x}{e^x+1}y=1.$

34. $3\frac{dy}{dx}-y\sec x=y^4\tan x.$

35. $\frac{dy}{dx}-2y=x^2.$

36. $\frac{dy}{dx}-ay=e^{ax}.$

37. $\frac{d^2y}{dx^2}=\sin ax.$

38. $\frac{d^2y}{dx^2}=xe^x.$

39. $\frac{d^2y}{dx^2}=x\log x.$

40. $\frac{d^2y}{dx^2}\frac{dy}{dx}+x=0.$

41. $x\frac{d^2y}{dx^2}-\frac{dy}{dx}=0.$

42. $(a-x)\frac{d^2y}{dx^2}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$

43. $(1+y)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0.$

44. $y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+\frac{dy}{dx}=0.$

45. $x\frac{d^2y}{dx^2}-3\frac{dy}{dx}=x+1.$

46. $\frac{d^2y}{dx^2}+y\frac{dy}{dx}=0.$

47. $x\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2-1=0.$

48. $2y\frac{d^2y}{dx^2}-2\left(\frac{dy}{dx}\right)^2=y^3.$

49. $\left(x\frac{d^2y}{dx^2}-\frac{dy}{dx}\right)\frac{dy}{dx}-1=0.$

50. $\frac{d^2y}{dx^2}=-k^2y.$

51. $\frac{d^2y}{dx^2}=k^2y.$

52. Solve $\frac{d^2y}{dx^2} = \frac{2}{3}y^2$, under the hypothesis that when $x = 1$, $y = 1$ and $\frac{dy}{dx} = \frac{2}{3}$.

53. Solve $\frac{d^2y}{dx^2} = \frac{1}{2}\sin 2y$, under the hypothesis that when $x = 1$, $y = \frac{\pi}{2}$ and $\frac{dy}{dx} = 1$.

54. Solve $\frac{d^2y}{dx^2} = \sin 2y$, under the hypothesis that when $y = 0$, $\frac{dy}{dx} = 0$.

55. Solve $2\frac{d^2y}{dx^2} = -\frac{k^2}{y^2}$, under the hypothesis that $\frac{dy}{dx} = 0$ when $y = \infty$.

56. Solve $\frac{d^2y}{dx^2} = y + 1$, under the hypothesis that when $x = 2$, $y = 0$ and $\frac{dy}{dx} = 1$.

57. Find the curve in which the slope of the tangent at any point is n times the slope of the straight line joining the point to the origin.

58. Find the curve in which the chain of a suspension bridge hangs, assuming that the load on the chain is proportional to its projection on a horizontal line.

59. Find the curve in which the angle between the radius vector and the tangent is n times the vectorial angle.

60. Find the curve such that the area included between the curve, the axis of x, a fixed ordinate, and a variable ordinate is proportional to the variable ordinate.

61. Show that, if the normal to a curve always passes through a fixed point, the curve is a circle.

62. Find the curve in which the length of the portion of the normal between the curve and the axis of x is proportional to the square of the ordinate.

63. Find the curve in which the perpendicular from the origin upon the tangent is equal to the abscissa of the point of contact.

64. Find the curve in which the perpendicular upon the tangent from the foot of the ordinate of the point of contact is constant.

65. Find the curve in which the length of the arc from a fixed point to any point P is proportional to the square root of the abscissa of P.

66. Find the curve in which the area bounded by the curve, the axis of x, a fixed ordinate, and a variable ordinate is proportional to the length of the arc which is part of the boundary.

67. Find the deflection of a beam fixed at one end and weighted at the other.

68. Find the deflection of a beam fixed at one end and uniformly loaded.

69. Find the deflection of a beam loaded at its center and supported at its ends.

70. Find the curve whose radius of curvature is constant.

71. Find the curve in which the radius of curvature at any point varies as the cube of the length of the normal between that point and the axis of x.

72. A particle moves in a straight line under the influence of an attracting force directed toward a fixed point on that line and varying as the distance from the point. Determine the motion.

73. A particle moves in a straight line under the influence of an attracting force directed toward a point on the line and varying inversely as the square of the distance from that point. Determine the motion.

74. Find the velocity acquired by a body sliding down a curve without friction, under the influence of gravity.

75. Assuming that gravity varies inversely as the square of the distance from the center of the earth, find the velocity acquired by a body falling from infinity to the surface of the earth.

CHAPTER IX

FUNCTIONS OF SEVERAL VARIABLES

83. Functions of more than one variable. *A quantity z is said to be a function of two variables, x and y, if the values of z are determined when the values of x and y are given.* This relation is expressed by the symbols $z = f(x, y)$, $z = F(x, y)$, etc.

Similarly, u is a function of three variables, x, y, and z, if the values of u are determined when the values of x, y, and z are given. This relation is expressed by the symbols $u = f(x, y, z)$, $u = F(x, y, z)$, etc.

Ex. 1. If r is the radius of the base of a circular cone, h its altitude, and v its volume, $v = \frac{1}{3}\pi r^2 h$, and v is a function of the two variables, r and h.

Ex. 2. If f denotes the centrifugal force of a mass m revolving with a velocity v in a circle of radius r, $f = \frac{mv^2}{r}$, and f is a function of the three variables, m, v, and r.

Ex. 3. Let v denote a volume of a perfect gas, t its absolute temperature, and p its pressure. Then $\frac{pv}{t} = k$, where k is a constant. This equation may be written in the three equivalent forms: $p = k\frac{t}{v}$, $v = k\frac{t}{p}$, $t = \frac{1}{k}pv$, by which each of the quantities, p, v, and t, is explicitly expressed as a function of the other two.

A function of a single variable is defined *explicitly* by the equation $y = f(x)$, and *implicitly* by the equation $F(x, y) = 0$. In either case the relation between x and y is represented graphically by a plane curve. Similarly, a function of two variables may be defined explicitly by the equation $z = f(x, y)$, or implicitly by the equation $F(x, y, z) = 0$. In either case the graphical representation of the function of two variables is the same, and may be made by introducing the conception of space coördinates.

84. Rectangular coördinates in space. To locate a point in space of three dimensions, we may assume three number scales, XX', YY', ZZ' (fig. 46), mutually perpendicular, and having their zero points coincident at O. They will determine three planes,

XOY, YOZ, ZOX, each of which is perpendicular to the other two. The planes are called the *coördinate planes*, and the three lines, XX', YY'; and ZZ', are called the axes of x, y, and z respectively, or the *coördinate axes*, and the point O is called the *origin of coördinates.*

Let P be any point in space, and through P pass planes perpendicular respectively to XX', YY', and ZZ', intersecting them at the points L, M, and N respectively. Then if we place $x = OL$, $y = OM$, and $z = ON$, as in I, § 16, it is evident that to any point there corresponds one, and only one, set of values of x, y, and z; and that to any set of values of x, y, and z there corresponds one, and only one, point. These values of x, y, and z are called the *coördinates* of the point, which is expressed as $P(x, y, z)$.

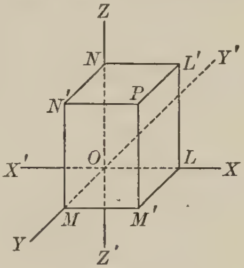

FIG. 46

From the definition of x it follows that x is equal, in magnitude and direction, to the distance of the point *from* the coördinate plane YOZ. Similar meanings are evident for y and z. It follows that a point may be plotted in several different ways by constructing in succession any three nonparallel edges of the parallelopiped (fig. 46) beginning at the origin and ending at the point.

In case the axes are not mutually perpendicular, we have a system of *oblique coördinates*. In this case the planes are passed through the point *parallel* to the coördinate planes. Then x gives the distance and the direction from the plane YOZ to the point, measured parallel to OX, and similar meanings are assigned to y and z. It follows that rectangular coördinates are a special case of oblique coördinates.

85. Graphical representation of a function of two variables. Let $f(x, y)$ be any function of two variables, and place

$$z = f(x, y). \tag{1}$$

Then the locus of all points the coördinates of which satisfy (1) is the graphical representation of the function $f(x, y)$. To construct this locus we may assign values to x and y, as $x = x_1$ and $y = y_1$, and compute from (1) the corresponding values of z. There will be, in general, distinct values of z, and if (1) defines an algebraic function, their number will be finite. The corresponding points all lie on a line parallel to OZ and intersecting XOY at the point $P_1(x_1, y_1)$, and these points alone of this line are points of the locus, and the portions of the line between them do not belong to the locus. As different values are assigned to x and y, new lines parallel to OZ are drawn on which there are, in general, isolated points of the locus. It follows that the locus has extension in only two dimensions, i.e. has no thickness, and is, accordingly, a surface. Therefore *the graphical representation of a function of two variables is a surface.**

If $f(x, y)$ is indeterminate for particular values of x and y, the corresponding line parallel to OZ lies entirely on the locus.

Since the equations $z = f(x, y)$ and $F(x, y, z) = 0$ are equivalent, and their graphical representations are the same, it follows that *the locus of any single equation in x, y, and z is a surface.*

There are apparent exceptions to the above theorem, if we demand that the surface shall have real existence. Thus, for example,

$$x^2 + y^2 + z^2 = -1$$

is satisfied by no real values of the coördinates. It is convenient in such cases, however, to speak of "imaginary surfaces."

Moreover, it may happen that the real coördinates which satisfy the equation may give points which lie upon a certain line, or are even isolated points. For example, the equation

$$x^2 + y^2 = 0$$

is satisfied in real coördinates only by the points $(0, 0, z)$ which lie upon the axis of z; while the equation

$$x^2 + y^2 + z^2 = 0$$

is satisfied, as far as real points go, only by $(0, 0, 0)$. In such cases it is still convenient to speak of a surface as represented by the equation, and to consider the part which may be actually constructed as the *real part* of that surface. The imaginary part is considered as made up of the points corresponding to sets of complex values of x, y, and z which satisfy the equation.

86. The appearance of any surface can be thus determined from its equation by assigning values to any two of the coördinates and computing the corresponding values of the third. This method, however, has practical difficulties. In place of it we may study any surface by means of the sections of the surface made by planes parallel to the coördinate planes. If, for example, we place $z = 0$ in the equation of any surface, the resulting equation in x and y is evidently the equation of the plane curve cut from the surface by the plane XOY. Again, if we place $z = z_1$, where z_1 is some fixed finite value, the resulting equation in x and y is the equation of the plane curve cut from the surface by a plane parallel to the plane XOY and z_1 units distant from it, and referred to new axes $O'X'$ and $O'Y'$, which are the intersections of the plane $z = z_1$ with the planes XOZ and YOZ respectively; for by placing $z = z_1$ instead of $z = 0$, we have virtually transferred the plane XOY, parallel to itself, through the distance z_1.

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* It is to be noted that this method of graphically representing a function cannot be extended to functions of more than two variables, since we have but three dimensions in space.

In applying this method it is advisable to find first the three plane sections made by the coördinate planes $x = 0$, $y = 0$, $z = 0$. These alone will sometimes give a general idea of the appearance of the surface, but it is usually desirable to study other plane sections on account of the additional information that may be derived.

The following surfaces have been chosen for illustration because it is important that the student should be familiar with them.

Ex. 1. $Ax + By + Cz + D = 0$.

Placing $z = 0$, we have (fig. 47)

$$Ax + By + D = 0. \qquad (1)$$

Hence the plane XOY cuts this surface in a straight line. Placing $y = 0$ and then $x = 0$, we find the sections of this surface made by the planes ZOX and YOZ to be respectively the straight lines

$$Ax + Cz + D = 0, \qquad (2)$$

and

$$By + Cz + D = 0. \qquad (3)$$

Placing $z = z_1$, we have

$$Ax + By + Cz_1 + D = 0, \qquad (4)$$

which is the equation of a straight line in the plane $z = z_1$. The line (4) is parallel to the line (1), since they make the angle $\tan^{-1}\left(-\frac{A}{B}\right)$ with the parallel lines $O'X'$ and OX and lie in parallel planes. To find the point where (4) intersects the plane XOZ, we place $y = 0$, and the result $Ax + Cz_1 + D = 0$ shows that this point is a point of the line (2). This result is true for all values of z_1. Hence this surface is the locus

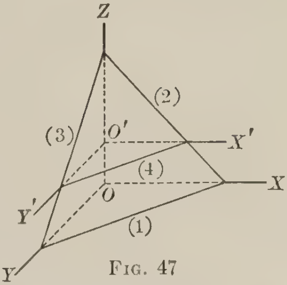

FIG. 47

of a straight line which moves along a fixed straight line always remaining parallel to a given initial position; hence it is a *plane*.

Since the equation $Ax + By + Cz + D = 0$ is the most general linear equation in three coördinates, we have proved that *the locus of every linear equation is a plane*.

Ex. 2. $z = ax^2 + by^2$, where $a > 0$, $b > 0$.

Placing $z = 0$, we have

$$ax^2 + by^2 = 0, \tag{1}$$

and hence the XOY plane cuts the surface in a point (fig. 48). Placing $y = 0$, we have

$$z = ax^2, \tag{2}$$

which is the equation of a parabola with its vertex at O and its axis along OZ. Placing $x = 0$, we have

$$z = by^2, \tag{3}$$

which is also the equation of a parabola with its vertex at O and its axis along OZ.

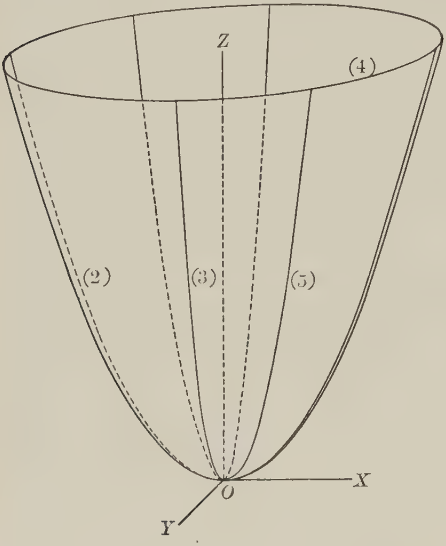

Fig. 48

Placing $z = z_1$, where $z_1 > 0$, we may write the resulting equation in the form

$$\frac{a}{z_1}x^2 + \frac{b}{z_1}y^2 = 1, \tag{4}$$

which is the equation of an ellipse with semiaxes $\sqrt{\frac{z_1}{a}}$ and $\sqrt{\frac{z_1}{b}}$. As the plane recedes from the origin, i.e. as z_1 increases, it is evident that the ellipse increases in magnitude.

If we place $z = -z_1$, the result may be written in the form $\frac{a}{z_1}x^2 + \frac{b}{z_1}y^2 = -1$, and hence there is no part of this surface on the negative side of the plane XOY.

The surface is called an *elliptic paraboloid*, and evidently may be generated by moving an ellipse of variable magnitude always parallel to the plane XOY, the ends of its axes always lying respectively on the parabolas $z = ax^2$ and $z = by^2$.

While the appearance of the surface is now completely determined, we shall, nevertheless, find it of interest to make two more sections. In the first place, we note that both the coördinate planes through OZ cut the surface in parabolas with their vertices at O and their axes along OZ, and hence arises the question, Do all planes through OZ cut the surface in parabolas?

To answer this question we shall make a transformation of coördinates by revolving the plane XOZ through an angle ϕ about OZ as an axis. The formulas of transformation will be

$$x = x' \cos\phi - y' \sin\phi,$$

$$y = x' \sin\phi + y' \cos\phi,$$

for z will not be changed at all, and x and y will be changed in the same manner as in I, § 115. The transformed equation is

$$z = a(x' \cos\phi - y' \sin\phi)^2 + b(x' \sin\phi + y' \cos\phi)^2.$$

Placing $y' = 0$, we have $$z = (a\cos^2\phi + b\sin^2\phi)\,x'^2, \tag{5}$$

which is a parabola with its vertex at O and its axis along OZ. But by changing ϕ we can make the plane $X'OZ$ any plane through OZ, and hence every plane through OZ cuts this surface in a parabola with its vertex at O and its axis along OZ.

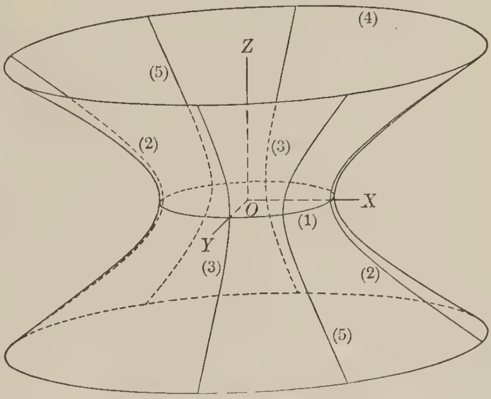

FIG. 49

Finally, we will place $y = y_1$ and write the resulting equation in the form

$$x^2 = \frac{1}{a}(z - by_1^2),$$

which is the equation of a parabola with its axis parallel to OZ and its vertex at a distance by_1^2 from the plane XOY.

Ex. 3. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$

Placing $z = 0$, we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{1}$$

which is the equation of an ellipse with semiaxes a and b (fig. 49). Placing $y = 0$, we have

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1, \tag{2}$$

which is the equation of an hyperbola with its transverse axis along OX and its conjugate axis along OZ. Placing $x = 0$, we have

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \tag{3}$$

which is the equation of an hyperbola with its transverse axis along OY and its conjugate axis along OZ.

If we place $z = \pm z_1$, and write the resulting equation in the form

$$\frac{x^2}{a^2\left(1 + \frac{z_1^2}{c^2}\right)} + \frac{y^2}{b^2\left(1 + \frac{z_1^2}{c^2}\right)} = 1, \tag{4}$$

we see that the section is an ellipse with semiaxes $a\sqrt{1 + \frac{z_1^2}{c^2}}$ and $b\sqrt{1 + \frac{z_1^2}{c^2}}$, which accordingly increases in magnitude as the cutting plane recedes from the origin, and that the surface is symmetrical with respect to the plane XOY, the result being independent of the sign of z_1.

Accordingly this surface, called the *unparted hyperboloid* or the *hyperboloid of one sheet*, may be generated by an ellipse of variable magnitude moving always parallel to the plane XOY and with the ends of its axes always lying on the hyperbolas $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$ and $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$.

If now we revolve the plane XOZ about OZ as an axis by making the transformation

$$x = x' \cos\phi - y' \sin\phi, \qquad y = x' \sin\phi + y' \cos\phi,$$

the transformed equation is

$$\frac{(x' \cos\phi - y' \sin\phi)^2}{a^2} + \frac{(x' \sin\phi + y' \cos\phi)^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Placing $y' = 0$, we have

$$\left(\frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2}\right)x'^2 - \frac{z^2}{c^2} = 1, \tag{5}$$

and have thus proved that the section made by any plane through OZ is an hyperbola with its conjugate axis along OZ.

Finally, we place $y = \pm y_1$, and write the resulting equation in the form

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{y_1^2}{b^2},$$

whence we see that the surface is symmetrical with respect to the plane XOZ. To discuss the equation further we shall have to make three cases according to the value of y_1.

If $y_1 < b$, we put the equation in the form

$$\frac{x^2}{a^2\left(1 - \frac{y_1^2}{b^2}\right)} - \frac{z^2}{c^2\left(1 - \frac{y_1^2}{b^2}\right)} = 1,$$

which is the equation of an hyperbola with its transverse axis equal to $2a\sqrt{1 - \frac{y_1^2}{b^2}}$ and parallel to OX. Hence the vertices will approach coincidence as the cutting plane recedes from the origin.

If $y_1 = b$, the equation may be put in the form

$$\left(\frac{x}{a} - \frac{z}{c}\right)\left(\frac{x}{a} + \frac{z}{c}\right) = 0,$$

which is the equation of two straight lines which intersect on OY.

If $y_1 > b$, we write the equation in the form

$$\frac{z^2}{c^2\left(\frac{y_1^2}{b^2} - 1\right)} - \frac{x^2}{a^2\left(\frac{y_1^2}{b^2} - 1\right)} = 1,$$

which is the equation of an hyperbola with its transverse axis equal to $2c\sqrt{\frac{y_1^2}{b^2} - 1}$ and parallel to OZ. Hence the vertices separate as the cutting plane recedes from the origin.

Ex. 4. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$

This surface (fig. 50) is a *cone*, with OZ as its axis and its vertex at O.

Ex. 5. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

This surface (fig. 51) is the *ellipsoid.*

Ex. 6. $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$

This surface (fig. 52) is the *biparted hyperboloid* or the *hyperboloid of two sheets.*

The discussions of the last three surfaces are very similar to that of the unparted hyperboloid, and for that reason they have been left to the student.

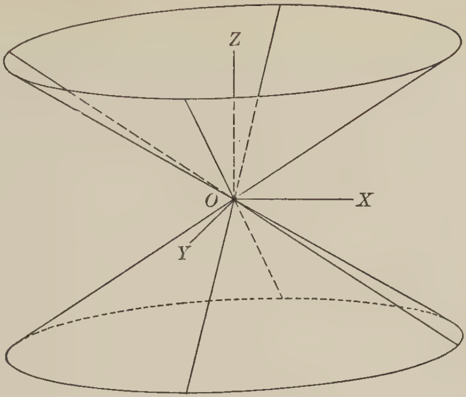

FIG. 50

Ex. 7. $z = ax^2 - by^2$, where $a > 0$, $b > 0$.

Placing $z = 0$, we obtain the equation

$$ax^2 - by^2 = 0, \tag{1}$$

i.e. two straight lines intersecting at the origin (fig. 53). Placing $y = 0$, we have

$$z = ax^2, \tag{2}$$

the equation of a parabola with its vertex at O and its axis along the positive direction of OZ.

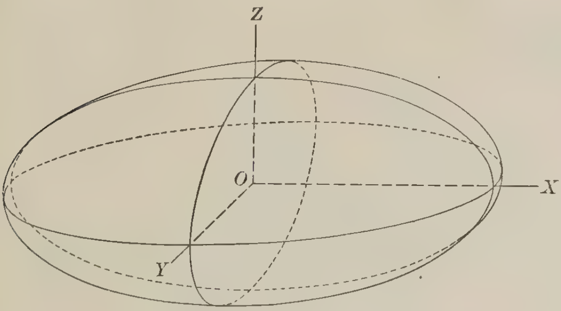

FIG. 51

Placing $x = 0$, we have

$$z = -by^2, \tag{3}$$

the equation of a parabola with its vertex at O and its axis along the negative direction of OZ.

Placing $x = \pm x_1$, we have

$$z = ax_1^2 - by^2,$$

or

$$y^2 = -\frac{1}{b}(z - ax_1^2), \tag{4}$$

a parabola with its axis parallel to OZ and its vertex at a distance ax_1^2 from the plane XOY. It is evident, moreover, that the surface is symmetrical with respect to the plane YOZ, and that the vertices of these parabolas, as different values are assigned to x_1, all lie on the parabola $z = ax^2$.

Hence this surface may be generated by the parabola $z = -by^2$ moving always parallel to the plane YOZ, its vertex lying on the parabola $z = ax^2$. The surface is called the *hyperbolic paraboloid.*

Finally, placing $z = z_1$, where $z_1 > 0$, we have

$$z_1 = ax^2 - by^2,$$

or $$\frac{a}{z_1}x^2 - \frac{b}{z_1}y^2 = 1, \quad (5)$$

an hyperbola with its transverse axis parallel to OX and increasing in length as the cutting plane recedes from the origin.

If $z = -z_1$, we may write the equation in the form

$$\frac{b}{z_1}y^2 - \frac{a}{z_1}x^2 = 1, \quad (6)$$

an hyperbola with its transverse axis parallel to OY and increasing in length as the cutting plane recedes from the origin.

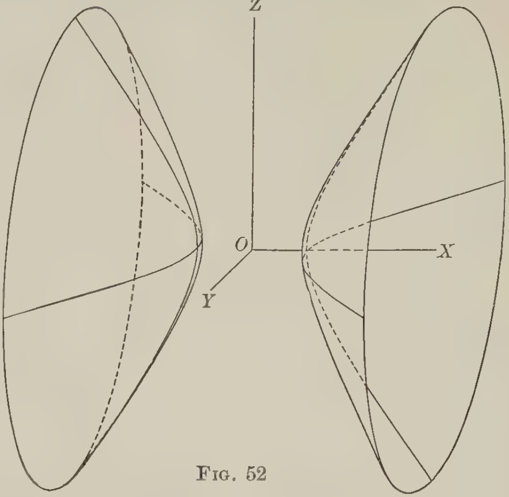

FIG. 52

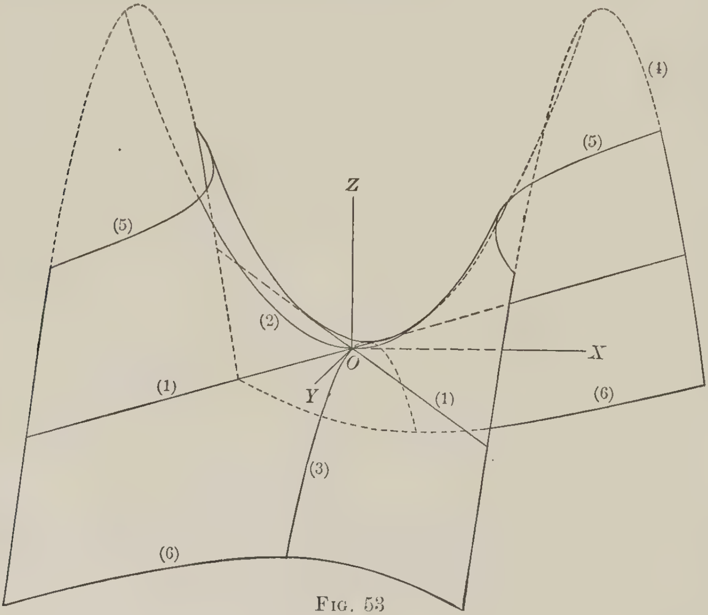

FIG. 53

Ex. 8. $z = kxy$.

As the algebraic sign of the constant k merely shows which side of the plane XOY we take as the positive side, we will assume $k > 0$, and discuss the surface (fig. 54) on that hypothesis.

Placing $z = 0$, we have $xy = 0$, which is the equation of the axes OX and OY. Placing $y = 0$, or $x = 0$, we have $z = 0$, and gain no new information about the surface.

Placing $z = z_1$, where $z_1 > 0$, we have

$$xy = \frac{z_1}{k}, \tag{1}$$

an hyperbola referred to its asymptotes as axes (I, § 117, Ex.). Placing $z = -z$, we have the hyperbola

$$xy = -\frac{z_1}{k}. \tag{2}$$

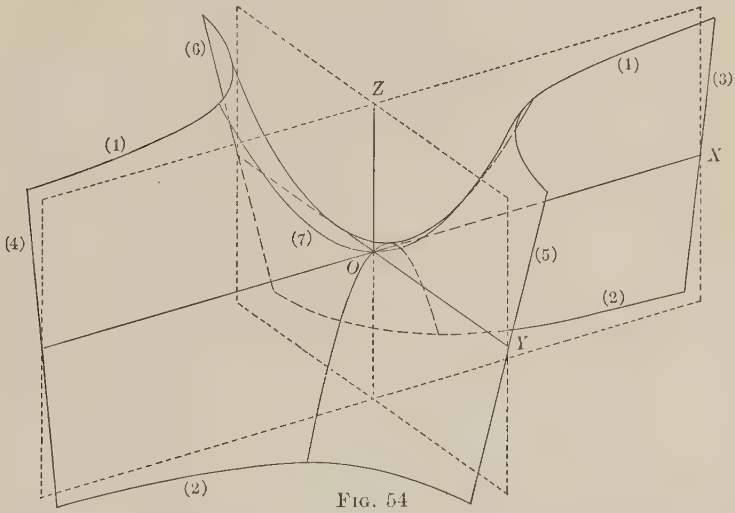

Fig. 54

Hence all sections made by planes parallel to XOY are hyperbolas with OX and OY as their asymptotes, and lying in the first and the third quadrants or in the second and the fourth quadrants according as the cutting plane is on the positive or the negative side of the XOY plane.

Placing $x = x_1$, where $x_1 > 0$, we have

$$z = kx_1y, \tag{3}$$

which is the equation of a straight line intersecting OX and parallel to the plane YOZ. If $x = -x_1$, we have the straight line

$$z = -kx_1y. \tag{4}$$

Similarly, placing $y = y_1$, where $y_1 > 0$, we have

$$z = ky_1x, \tag{5}$$

which is the equation of a straight line intersecting OY and parallel to the plane ZOX. If $y = -y_1$, we have the straight line

$$z = -ky_1x. \tag{6}$$

As the cutting plane recedes from the origin it is evident that these straight lines revolve about OX and OY respectively as they move along them, but always remain parallel to the planes YOZ and ZOX respectively.

Finally, we will revolve the plane XOZ about OZ as an axis, the transformed equation being

$$z = k(x' \cos\phi - y' \sin\phi)(x' \sin\phi + y' \cos\phi).$$

Placing $y' = 0$, we have

$$z = kx'^2 \cos\phi \sin\phi, \tag{7}$$

the equation of a parabola with its vertex at O and its axis along OZ. Hence all planes through OZ cut this surface in parabolas with their axes along OZ.

The surface is a special case of the *hyperbolic paraboloid* of Ex. 7; for if we keep OZ in its original position and swing OX and OY into new positions by the formulas of I, § 117, and choose the angle ϕ as in the illustrative example of that article, the equation $z = ax^2 - by^2$ assumes the form $z = \frac{4ab}{a+b}xy$. Here the coördinates are oblique unless $b = a$; but if $b = a$, the coördinates are rectangular and we have the case just considered, where $k = 2a$.

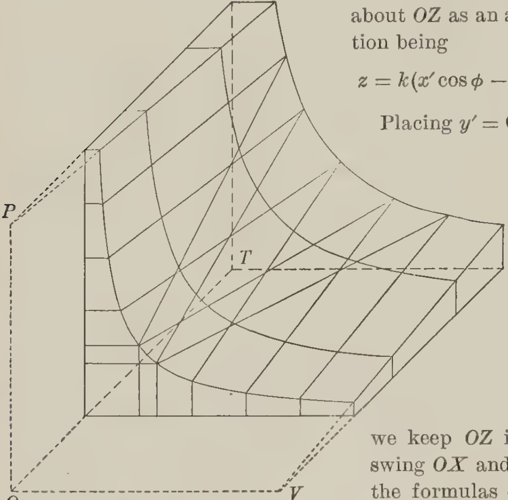

FIG. 55

The portion of this surface on which the coördinates are all positive shows graphically the relations between the pressure, the volume, and the temperature of a perfect gas (§ 83, Ex. 3). This part of the surface is shown in fig. 55.

87. Surfaces of revolution. If the sections of a surface made by planes parallel to one of the coördinate planes are circles with their centers on the axis of coördinates which is perpendicular to the cutting planes, the surface is a *surface of revolution* (§ 37) with that coördinate axis as the axis of revolution. This will always occur when the equation of the surface is in the form $F(\sqrt{x^2+y^2}, z) = 0$, which means that the two coördinates x and y enter only in the combination $\sqrt{x^2+y^2}$; for if we place $z = z_1$ in this equation to find the corresponding section, and solve the resulting equation for $x^2 + y^2$, we have, as a result, the equation of one or more circles, according to the number of roots of the equation in $x^2 + y^2$.

Again, if we place $x = 0$, we have the equation $F(y, z) = 0$, which is the equation of the generating curve in the plane YOZ. Similarly, if we place $y = 0$, we have $F(x, z) = 0$, which is the equation of the generating curve in the plane XOZ. It should be noted that the coördinate which appears uniquely in the equation shows which axis of coördinates is the axis of revolution.

Conversely, if we have any plane curve $F(x, z) = 0$ in the plane XOZ, the equation of the surface formed by revolving it about OZ as an axis is $F(\sqrt{x^2 + y^2}, z) = 0$, which is formed by simply replacing x in the equation of the curve by $\sqrt{x^2 + y^2}$.

Ex. 1. Show that the unparted hyperboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{a^2} = 1$ is a surface of revolution.

Writing this equation in the form

$$\frac{x^2 + z^2}{a^2} - \frac{y^2}{b^2} - 1 = 0,$$

we see that it is a surface of revolution with OY as the axis.

Placing $z = 0$, we have $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, an hyperbola, as the generating curve. The hyperbola was revolved about its conjugate axis.

Ex. 2. Find the paraboloid of revolution generated by revolving the parabola $y^2 = 4px$ about its axis.

Replacing y by $\sqrt{y^2 + z^2}$, we have as the equation of the required surface $y^2 + z^2 = 4px$.

88. Cylinders. If a given equation is of the form $F(x, y) = 0$, involving only two of the coördinates, it might appear to represent a curve lying in the plane of those coördinates. But if we are dealing with space of three dimensions, such an interpretation would be incorrect, in that it amounts to restricting z to the value $z = 0$, whereas, in fact, the value of z corresponding to any simultaneous values of x and y satisfying the equation $F(x, y) = 0$ may be anything whatever. Hence, corresponding to every point of the curve $F(x, y) = 0$ in the plane XOY, there is an entire straight line, parallel to OZ, on the surface $F(x, y) = 0$. Such a surface is a *cylinder*, its directrix being the plane curve $F(x, y)=0$ in the plane $z=0$, and its elements being parallel to OZ, the axis of the coördinate not present.

For example, $x^2+y^2=a^2$ is the equation of a circular cylinder, its elements being parallel to OZ, and its directrix being the circle $x^2+y^2=a^2$ in the plane XOY.

In like manner, $z^2=ky$ is a parabolic cylinder with its elements parallel to OX.

If only one coördinate is present in the equation, the locus is a number of planes. For example, the equation $x^2-(a+b)x+ab=0$ may be written in the form $(x-a)(x-b)=0$, which represents the two planes $x-a=0$ and $x-b=0$. Similarly, any equation involving only one coördinate determines values of that coördinate only and the locus is a number of planes.

Regarding a plane as a cylinder of which the directrix is a straight line, we may say that *any equation not containing all the coördinates represents a cylinder.*

If the axes are oblique, the elements of the cylinders are not perpendicular to the plane of the directrix.

89. Space curves. The two surfaces represented respectively by the equations $f_1(x, y, z)=0$ and $f_2(x, y, z)=0$ intersect, in general, in a curve, the coördinates of every point of which satisfy each of the equations. Conversely, any point the coördinates of which satisfy these equations simultaneously is in their curve of intersection. Hence, *in general, the locus of two simultaneous equations in x, y, and z is a curve.*

In particular, *the locus of the two simultaneous linear equations,*

$$A_1x+B_1y+C_1z+D_1=0,$$
$$A_2x+B_2y+C_2z+D_2=0,$$

is a straight line, since it is the line of intersection of the two planes respectively represented by the two equations.

If, in the equations $f_1(x, y, z)=0$, $f_2(x, y, z)=0$, we assign a value to one of the coördinates, as x for example, there are two equations from which to determine the corresponding values of y and z, in general a determinate problem. But if values are assigned to two of the coördinates, as x and y, there are two equations from which to determine a single unknown z, a problem generally impossible. Hence there is only one independent variable in the equations of a curve.

In general, we may make x the independent variable and place the equations of the curve in the form $y=\phi_1(x)$, $z=\phi_2(x)$, by solving the original equations of the curve for y and z in terms of x. The new surfaces, $y=\phi_1(x)$, $z=\phi_2(x)$, determining the curve, are cylinders (§ 88), with elements parallel to OZ and OY respectively. The equation $y=\phi_1(x)$ interpreted in the plane XOY is the equation of the projection (§ 92) of the curve on that plane. Similarly, the equation $z=\phi_2(x)$, interpreted in the plane ZOX, is the equation of the projection of the curve on that plane.

In particular, if we solve the equations

$$A_1x+B_1y+C_1z+D_1=0,$$
$$A_2x+B_2y+C_2z+D_2=0,$$

for y and z in terms of x, we have two equations of the form

$$y=px+q, \qquad z=rx+s,$$

as the equations of the same straight line that was represented by the original equations.

90. If t is any variable parameter, and we make x a function of t, as $x=f_1(t)$, and substitute this value of x in the equations $y=\phi_1(x), z=\phi_2(x)$ (§ 89), we have

$$x=f_1(t), \qquad y=f_2(t), \qquad z=f_3(t),$$

as the parametric equations of the curve.

More generally, the three equations

$$x=f_1(t), \qquad y=f_2(t), \qquad z=f_3(t),$$

represent a curve, the equations of which may be generally put in the form $y=\phi_1(x), z=\phi_2(x)$, by eliminating t from the first and the second equations, and from the first and the third equations.

Ex. The space curve, called the *helix*, is the path of a point which moves around the surface of a right circular cylinder with a constant angular velocity and at the same time moves parallel to the axis of the cylinder with a constant linear velocity.

Let the radius of the cylinder (fig. 56) be a, and let its axis coincide with OZ. Let the constant angular velocity be ω and the constant linear velocity be v. Then if θ denotes the angle through which the plane ZOP has swung from its initial position ZOX, the coördinates of any point $P(x, y, z)$ of the helix are given by the equations

$$x = a\cos\theta, \quad y = a\sin\theta, \quad z = vt.$$

But $\theta = \omega t$, and accordingly we may have as the parametric equations of the helix

$$x = a\cos\omega t, \quad y = a\sin\omega t, \quad z = vt,$$

t being the variable parameter.

Or, since $t = \frac{\theta}{\omega}$, we may regard θ as the variable parameter, and the equations are

$$x = a\cos\theta, \quad y = a\sin\theta, \quad z = k\theta,$$

where k is the constant $\frac{v}{\omega}\cdot$

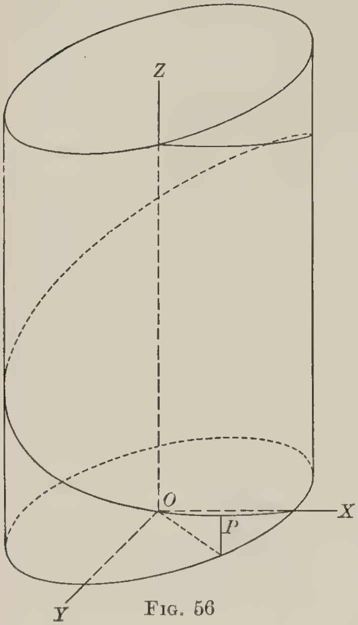

FIG. 56

91. Ruled surfaces. A surface which may be generated by a moving straight line is called a *ruled surface.* The plane, the cone, and the cylinder are simple examples of ruled surfaces, and in § 86, Ex. 8, it was shown that the hyperbolic paraboloid is a ruled surface.

Ex. 1. Prove that the ruled surface generated by a straight line which moves so as to intersect two fixed straight lines not in the same plane and at the same time remain parallel to a fixed plane is an hyperbolic paraboloid.

Let the fixed straight lines have the equations $y = 0$, $z = 0$, and $y = c$, $z = kx$, and let YOZ be the fixed plane, the axes of coördinates being oblique. Then the straight line $x = a$, $y = mz$ is a straight line which is parallel to the plane YOZ and intersects the line $y = 0$, $z = 0$, for all values of a and m. If this line intersects the line $y = c$, $z = kx$, evidently $m = \frac{c}{ka}\cdot$ Therefore the equations of the required line are $x = a$, $y = \frac{c}{ka}z$. Any values of x, y, z which satisfy these equations satisfy their product $xy = \frac{c}{k}z$, or $z = \frac{k}{c}xy$. Hence the line always lies on the surface $z = \frac{k}{c}xy$, an hyperbolic paraboloid (§ 86, Ex. 8).

Ex. 2. Prove that the unparted hyperboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ is a ruled surface having two sets of *rectilinear generators*, i.e. that through every point of it two straight lines may be drawn, each of which shall lie entirely on the surface.

If we write the equation of the hyperboloid in the form

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{y^2}{b^2}, \tag{1}$$

it is evident that (1) is the product of the two equations

$$\begin{aligned} \frac{x}{a} - \frac{z}{c} &= k_1\left(1 - \frac{y}{b}\right), \\ \frac{x}{a} + \frac{z}{c} &= \frac{1}{k_1}\left(1 + \frac{y}{b}\right), \end{aligned} \tag{2}$$

for any value of k_1. But (2) are the equations of a straight line (§ 89). Moreover, this straight line lies entirely on the surface, since the coördinates of every point of it satisfy (2) and hence (1). As different values are assigned to k_1, we obtain a series of straight lines lying entirely on the surface.

Conversely, if $P_1(x_1, y_1, z_1)$ is any point of (1),

$$\frac{\dfrac{x_1}{a} - \dfrac{z_1}{c}}{1 - \dfrac{y_1}{b}} = \frac{1 + \dfrac{y_1}{b}}{\dfrac{x_1}{a} + \dfrac{z_1}{c}}.$$

Therefore P_1 determines the same value of k_1 from both equations (2). Hence every point of (1) lies in one and only one line (2).

We may also regard (1) as the product of the two equations

$$\begin{aligned} \frac{x}{a} - \frac{z}{c} &= k_2\left(1 + \frac{y}{b}\right), \\ \frac{x}{a} + \frac{z}{c} &= \frac{1}{k_2}\left(1 - \frac{y}{b}\right), \end{aligned} \tag{3}$$

whence it is evident that there is a second set of straight lines lying entirely on the surface, one and only one of which may be drawn through any point of the surface.

Equations (2) and (3) are the equations of the rectilinear generators, and every point of the surface may be regarded as the point of intersection of one line from each set.

Ex. 3. The *conoid* (Ex. 52, p. 83) is evidently a ruled surface, since it is generated by a straight line moving always parallel to a given plane and at the same time intersecting a given curve and a given straight line.

Ex. 4. Another example of a ruled surface is the *cylindroid*, i.e. the surface generated by a straight line moving parallel to a given plane and touching two given curves.

Ex. 5. Show that the prismoidal formula (§ 39) applies to the volume bounded by a ruled surface and two parallel planes.

The equations $x = pz + q$, $y = rz + s$ represent a straight line. If p, q, r, and s are functions of a single independent parameter t, the line generates a ruled surface.

If we place $z = 0$, we have $x = q$, $y = s$, the parametric equations of the section of the ruled surface by the plane XOY. Similarly, if we place $z = z_1$, we have $x = pz_1 + q$, $y = rz_1 + s$, the parametric equations of a section parallel to XOY. Suppose now when t varies from t_0 to t_1, the straight-line generator of the surface traverses the perimeter of the section $z = z_1$. Then the area of this section is

$$\begin{aligned}\int_{t_0}^{t_1} y\,dx &= \int_{t_0}^{t_1}(rz_1 + s)\left(\frac{dp}{dt}z_1 + \frac{dq}{dt}\right)dt\\ &= z_1^2\int_{t_0}^{t_1} r\frac{dp}{dt}\,dt + z_1\int_{t_0}^{t_1}\left(s\frac{dp}{dt} + r\frac{dq}{dt}\right)dt + \int_{t_0}^{t_1} s\frac{dq}{dt}\,dt.\end{aligned}$$

This is a quadratic polynomial in z_1. Hence the prismoidal formula holds for a portion of a ruled surface bounded by parallel planes.

The prismoid itself is a special case of such a surface.

PROBLEMS

1. Show that the surface $lf_1(x, y, z) + kf_2(x, y, z) = 0$, where l and k are constants, is a surface passing through all the points common to the two surfaces $f_1(x, y, z) = 0$ and $f_2(x, y, z) = 0$, and meeting them at no other points.

2. Discuss the surface $xyz = a^3$ by means of plane sections.

3. Show that the surface $z = a - \sqrt{x^2 + y^2}$ is a cone of revolution, and find its vertex and axis.

4. Prove that the surface $ax + by = cz^2$ is a cylindroid, and discuss its plane sections.

5. Discuss the surface $x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}} = a^{\frac{1}{2}}$ by means of plane sections.

6. Prove that the surface $(ax + by)^2 = cz$ is a cylindroid, and discuss its plane sections.

7. Discuss the surface $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}$ by means of plane sections.

8. Find the equation of a right circular cylinder which is tangent to the plane XOZ and has its axis in the plane YOZ.

9. Find the equation of a right circular cylinder which is tangent to the planes XOZ and XOY.

10. Describe the locus of the equation $z + (y + 1)^2 = 0$.

11. Describe the locus of the equation $2y^2 - 5y - 3 = 0$.

12. Describe the locus of the equation $(ax + by)^2 - c^2z^2 = 0$.

13. Find the equation of the *oblate spheroid*, i.e. the surface generated by an ellipse revolving about its minor axis.

14. Find the equation of a biparted hyperboloid of revolution with OY as its axis.

15. Find the equation of the *prolate spheroid*, i.e. the surface generated by revolving an ellipse about its major axis.

16. Find the equation of the surface generated by revolving the parabola $y^2 = 4px$ about OY as an axis.

17. Find the equation of the ring surface generated by revolving about OX the circle $x^2 + (y - b)^2 = a^2$, where $a < b$.

18. Write the equation of the surface generated by revolving the hyperbola $xy = c$ about either of its asymptotes as an axis.

19. Find the equation of the surface formed by revolving the four-cusped hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about OY as an axis.

20. What surface is represented by the equation $x^{\frac{1}{2}} + (y^2 + z^2)^{\frac{1}{4}} = a^{\frac{1}{2}}$?

21. What surface is represented by the equation $x^2 + z^2 - y^6 = 0$?

22. What is the line represented by the equations $y^2 + z^2 - 6x = 0$, $x - 3 = 0$?

23. Show that the line of intersection of the surfaces $x^2 + y^2 = a^2$ and $y = z$ is an ellipse. (Rotate the axes about OX through 45°.)

24. Show that the projections of the skew cubic $x = t$, $y = t^2$, $z = t^3$ on the coördinate planes are a parabola, a semicubical parabola, and a cubical parabola.

25. Prove that the projections of the helix $x = a \cos \theta$, $y = a \sin \theta$, $z = k\theta$ on the planes XOZ and YOZ are sine curves, the width of each arch of which is $k\pi$.

26. Prove that the projection of the curve $x = e^t$, $y = e^{-t}$, $z = t\sqrt{2}$, on the plane XOY is an equilateral hyperbola.

27. Turn the plane XOZ about OZ as an axis through an angle of 45°, and show that the projection of the curve $x = e^t$, $y = e^{-t}$, $z = t\sqrt{2}$ on the new XOZ plane is a catenary.

28. Show that the curve $x = t^2$, $y = 2t$, $z = t$ is a plane section of a parabolic cylinder.

29. Prove that the skew quartic $x = t$, $y = t^3$, $z = t^4$ is the intersection of an hyperbolic paraboloid and a cylinder the directrix of which is a cubical parabola.

30. The vertical angle of a cone of revolution is 90°, its vertex is at O, and its axis coincides with OZ. A point, starting from the vertex, moves in a spiral path along the surface of the cone so that the measure of the distance it has traveled parallel to the axis of the cone is equal to the circular measure of the angle through which it has revolved about the axis of the cone. Prove that the equations of its path, called the *conical helix*, are $x = t \cos t$, $y = t \sin t$, $z = t$.

CHAPTER X

PLANE AND STRAIGHT LINE

92. Projection. The *projection* of a point on a straight line is defined as the point of intersection of the line and a plane through the point perpendicular to the line. Hence in fig. 46 L, M, and N are the projections of the point P on the axes of x, y, and z respectively.

The projection of one straight line of finite length upon a second straight line is the part of the second line included between the projections of the ends of the first line, its direction being from the projection of the initial point of the first line to the projection of the terminal point of the first line. In fig. 57, for example, the projections of A and B on MN being A' and B' respectively, the projection of AB on MN is $A'B'$, and the projection of BA on MN is $B'A'$. If MN and AB denote the positive directions respectively of these lines, it follows that $A'B'$ is positive when it has the same direction as MN and is negative when it has the opposite direction to MN.

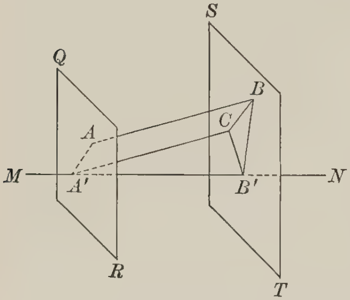

FIG. 57

In particular, the projection on OX of the straight line P_1P_2 drawn from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is L_1L_2, where $OL_1 = x_1$ and $OL_2 = x_2$. But $L_1L_2 = x_2 - x_1$ by I, § 13. Hence the projection of P_1P_2 on OX is $x_2 - x_1$; and similarly, its projections on OY and OZ are respectively $y_2 - y_1$ and $z_2 - z_1$.

If we define the angle between any two lines in space as the angle between lines parallel to them and drawn from a common point, then *the projection of one straight line on a second is the* *product of the length of the first line and the cosine of the angle between the positive directions of the two lines.* Then if ϕ is the angle between AB and MN (fig. 57),

$$A'B' = AB \cos \phi.$$

To prove this proposition, draw $A'C$ parallel to AB and meeting the plane ST at C. Then $A'C = AB$, and $A'B' = A'C \cos \phi$, by I, § 14, whence the truth of the proposition is evident.

Defining the projection of a broken line upon a straight line as the sum of the projections of its segments, we may prove, as in I, § 15, that *the projections on any straight line of a broken line and the straight line joining its ends are the same.*

We will now show that *the projection of any plane area upon another plane is the product of that area and the cosine of the angle between the planes.*

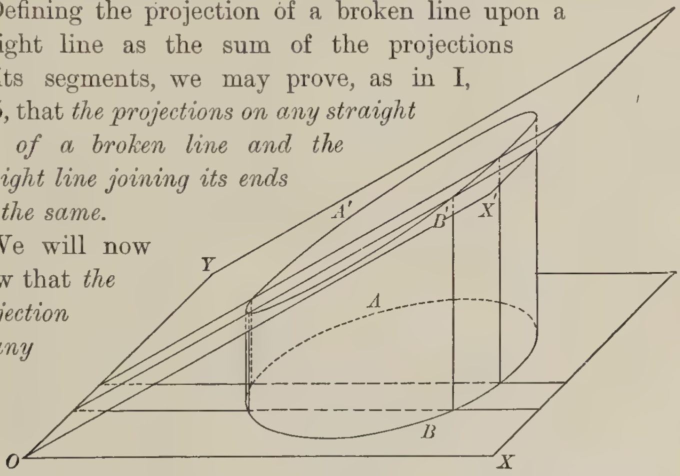

Fig. 58

Let $X'OY$ (fig. 58) be any plane through OY making an angle ϕ with the plane XOY. Let $A'B'$ be any area in $X'OY$ such that any straight line parallel to OX' intersects its boundary in not more than two points, and let AB be its projection on XOY.

Then (§ 35)
$$\text{area } A'B' = \int (x_2' - x_1')\, dy, \tag{1}$$

the limits of integration being taken so as to include the whole area.

In like manner,
$$\text{area } AB = \int (x_2 - x_1)\, dy, \tag{2}$$

the limits of integration being taken so as to cover the whole area.

But the values of y are the same in both planes, since they are measured parallel to the line of intersection of the two planes; and hence the limits in (1) and (2) are the same. Since the x coördinate is measured perpendicular to the line of intersection, $x_2 = x'_2 \cos\phi$, $x_1 = x'_1 \cos\phi$, and (2) becomes

$$\begin{aligned} \text{area } AB &= \int (x'_2 - x'_1) \cos\phi \, dy \\ &= \cos\phi \int (x'_2 - x'_1)\, dy \\ &= (\cos\phi)(\text{area } A'B'). \end{aligned}$$

93. Distance between two points. Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ (fig. 59) be any two points. Pass planes through P_1 and P_2 parallel to the coördinate planes, thereby forming a rectangular parallelopiped having the edges parallel to OX, OY, and OZ, and having P_1P_2 as a diagonal. Then the edges are equal respectively to $x_2 - x_1$, $y_2 - y_1$, and $z_2 - z_1$ (§ 92), being equal to the projections of P_1P_2 on OX, OY, and OZ.

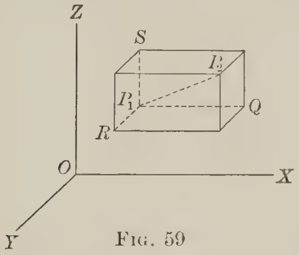

FIG. 59

Hence

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

If the two points have two coördinates the same, as, for example, $y_2 = y_1$, $z_2 = z_1$, the formula reduces to $P_1P_2 = x_2 - x_1$.

Ex. 1. Find a point $\sqrt{14}$ units distant from each of the three points $(1, 0, 3)$, $(2, -1, 1)$, $(3, 1, 2)$.

Let $P(x, y, z)$ be the required point.

Then
$$\begin{aligned} (x-1)^2 + (y-0)^2 + (z-3)^2 &= 14, \\ (x-2)^2 + (y+1)^2 + (z-1)^2 &= 14, \\ (x-3)^2 + (y-1)^2 + (z-2)^2 &= 14. \end{aligned}$$

Solving these three equations, we determine the two points $(0, 2, 0)$ and $(4, -2, 4)$.

Ex. 2. Find the equation of a sphere of radius r with its center at $P_1(x_1, y_1, z_1)$.

If $P(x, y, z)$ is any point of the sphere,

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2. \tag{1}$$

Conversely, if $P(x, y, z)$ is any point the coördinates of which satisfy (1), P is at the distance r from P_1, and hence is a point of the sphere. Therefore (1) is the required equation of the sphere.

94. Length of a space curve in rectangular coördinates. The method of finding the length of a space curve is similar to that of finding the length of a plane curve, so that the proof of § 40 may be repeated.

If AB (fig. 60) is the given curve, we assume $n-1$ points $P_1, P_2, P_3, \cdots, P_{n-1}$ between A and B, and connect each pair of consecutive points by a straight line. The length AB is then defined as the limit of the sum of the lengths of the n chords AP_1, $P_1P_2, \cdots, P_{n-1}B$, as n is increased without limit and the length of each chord approaches zero as a limit (I, § 104).

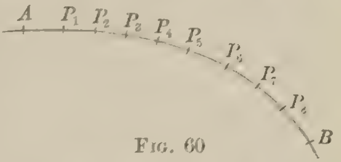

FIG. 60

Let the coördinates of P_i be (x_i, y_i, z_i) and those of P_{i+1} be $(x_i + \Delta x, y_i + \Delta y, z_i + \Delta z)$. Then $P_iP_{i+1} = \sqrt{\overline{\Delta x}^2 + \overline{\Delta y}^2 + \overline{\Delta z}^2}$ (§ 93). Now if x, y, z are functions of a variable parameter t (§ 90), and have the derivatives $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, then $\sqrt{\overline{\Delta x}^2 + \overline{\Delta y}^2 + \overline{\Delta z}^2}$ is an infinitesimal which differs from the infinitesimal $\sqrt{dx^2 + dy^2 + dz^2}$ by an infinitesimal of higher order. For we have

$$\frac{\sqrt{\overline{\Delta x}^2 + \overline{\Delta y}^2 + \overline{\Delta z}^2}}{\sqrt{dx^2 + dy^2 + dz^2}} = \frac{\sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2}}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}} \cdot \frac{\Delta t}{dt}.$$

Since t is the independent variable, $\Delta t = dt$ (§ 4); also

$$\operatorname*{Lim}_{\Delta t \doteq 0} \left(\frac{\Delta x}{\Delta t}\right) = \frac{dx}{dt}, \text{ etc.}$$

Hence
$$\text{Lim} \frac{\sqrt{\overline{\Delta x}^2 + \overline{\Delta y}^2 + \overline{\Delta z}^2}}{\sqrt{dx^2 + dy^2 + dz^2}} = 1.$$

Therefore (§ 3), in finding the length of the curve, we may replace $\sqrt{\overline{\Delta x}^2 + \overline{\Delta y}^2 + \overline{\Delta z}^2}$ by $\sqrt{dx^2 + dy^2 + dz^2}$. Hence, if s denotes the length of AB, we have

$$s = \int_{(A)}^{(B)} \sqrt{dx^2 + dy^2 + dz^2},$$

where (A) and (B) denote the values of the independent variable for the points A and B respectively. From this formula for s it follows (§ 9) that

$$ds = \sqrt{dx^2 + dy^2 + dz^2}.$$

Ex. Find the length of an arc of the helix $x = a\cos\theta$, $y = a\sin\theta$, $z = k\theta$, corresponding to an increase of 2π in θ.

Here $dx = -a\sin\theta\, d\theta$, $dy = a\cos\theta\, d\theta$, and $dz = k\, d\theta$.

Therefore

$$\begin{aligned} s &= \int_{\theta_1}^{\theta_1 + 2\pi} \sqrt{a^2 + k^2}\, d\theta \\ &= 2\pi\sqrt{a^2 + k^2}. \end{aligned}$$

95. Direction of a straight line. The direction of any straight line in space is determined by means of the angles which it makes with the positive directions of the coördinate axes OX, OY, OZ. We denote these angles by α, β, and γ respectively (fig. 61). Then their cosines, i.e. $\cos\alpha$, $\cos\beta$, $\cos\gamma$, are called the *direction cosines* of the line.

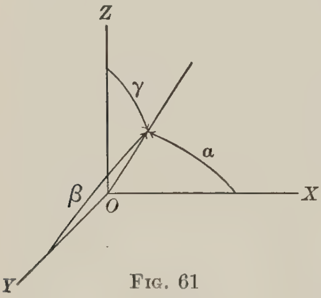

FIG. 61

It is to be noted that the same straight line makes the angles α_1, β_1, γ_1 or $\pi - \alpha_1$, $\pi - \beta_1$, $\pi - \gamma_1$ with the coördinate axes, according to the direction in which the line is drawn. Hence its direction cosines are either $\cos\alpha_1$, $\cos\beta_1$, $\cos\gamma_1$ or $-\cos\alpha_1$, $-\cos\beta_1$, $-\cos\gamma_1$. A straight line in which the direction is fixed has only one set of direction cosines.

The three direction cosines are not independent, for they satisfy the equation

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1;$$

that is, *the sum of the squares of the direction cosines of any straight line is always equal to unity.*

To prove this theorem, let the line P_1P_2 (fig. 59) be any straight line and let it make the angles α, β, and γ with OX, OY, and OZ respectively. Then

$$\cos\alpha = \frac{P_1Q}{P_1P_2}, \qquad \cos\beta = \frac{P_1R}{P_1P_2}, \qquad \cos\gamma = \frac{P_1S}{P_1P_2}.$$

Squaring and adding these equations, we have

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{\overline{P_1Q}^2 + \overline{P_1R}^2 + \overline{P_1S}^2}{\overline{P_1P_2}^2}.$$

But $$\overline{P_1Q}^2 + \overline{P_1R}^2 + \overline{P_1S}^2 = \overline{P_1P_2}^2.$$

Therefore $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$

96. If $P_1(x_1, y_1, z_1)$ is any fixed point (fig. 59), the coördinates of any second point P_2 may be denoted by $(x_1 + \Delta x, y_1 + \Delta y, z_1 + \Delta z)$, where Δx, Δy, and Δz are arbitrary.

If $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the straight line P_1P_2, we have, by the work of the last article,

$$\cos\alpha = \frac{\Delta x}{\sqrt{\overline{\Delta x}^2 + \overline{\Delta y}^2 + \overline{\Delta z}^2}}, \qquad \cos\beta = \frac{\Delta y}{\sqrt{\overline{\Delta x}^2 + \overline{\Delta y}^2 + \overline{\Delta z}^2}},$$

$$\cos\gamma = \frac{\Delta z}{\sqrt{\overline{\Delta x}^2 + \overline{\Delta y}^2 + \overline{\Delta z}^2}},$$

for $$P_1Q = \Delta x, \qquad P_1R = \Delta y, \qquad P_1S = \Delta z,$$

and $$P_1P_2 = \sqrt{\overline{\Delta x}^2 + \overline{\Delta y}^2 + \overline{\Delta z}^2}.$$

Hence, if Δx, Δy, and Δz are given, the direction of a straight line is determined, but the particular straight line having that direction is not determined, for the values of Δx, Δy, and Δz in no way determine x_1, y_1, and z_1, the coördinates of the initial point of the line. Moreover, the ratios of Δx, Δy, and Δz are alone essential in determining the direction of the line. Accordingly we may speak of the direction $\Delta x : \Delta y : \Delta z$, meaning thereby the direction of a straight line, the direction cosines of which are respectively

$$\frac{\Delta x}{\sqrt{\overline{\Delta x}^2 + \overline{\Delta y}^2 + \overline{\Delta z}^2}}, \quad \frac{\Delta y}{\sqrt{\overline{\Delta x}^2 + \overline{\Delta y}^2 + \overline{\Delta z}^2}}, \quad \frac{\Delta z}{\sqrt{\overline{\Delta x}^2 + \overline{\Delta y}^2 + \overline{\Delta z}^2}}.$$

Furthermore, if A, B, and C are any three given numbers, we may speak of the direction $A : B : C$. For we may place $\Delta x = A$, $\Delta y = B$, $\Delta z = C$, and thus determine a direction, the direction cosines of which are

$$\frac{A}{\sqrt{A^2 + B^2 + C^2}}, \quad \frac{B}{\sqrt{A^2 + B^2 + C^2}}, \quad \frac{C}{\sqrt{A^2 + B^2 + C^2}}.$$

97. Direction of a space curve. If we regard the position of a point $P(x, y, z)$ of a curve as determined by s, — its distance along the curve from some fixed point of the curve, — as in I, § 105, x, y, and z are functions of s, i.e.

$$x = f_1(s), \qquad y = f_2(s), \qquad z = f_3(s).$$

Then if s is increased by an increment Δs, a second point $Q(x+\Delta x, y+\Delta y, z+\Delta z)$ of the curve is located, and the direction cosines of the chord PQ are

$$\frac{\Delta x}{\sqrt{\overline{\Delta x}^2+\overline{\Delta y}^2+\overline{\Delta z}^2}}, \quad \frac{\Delta y}{\sqrt{\overline{\Delta x}^2+\overline{\Delta y}^2+\overline{\Delta z}^2}}, \quad \frac{\Delta z}{\sqrt{\overline{\Delta x}^2+\overline{\Delta y}^2+\overline{\Delta z}^2}}. \qquad (\S\ 96)$$

The limits of these ratios, as $\Delta s \doteq 0$, are the direction cosines of the tangent to the curve at P.

Now
$$\frac{\Delta x}{\sqrt{\overline{\Delta x}^2+\overline{\Delta y}^2+\overline{\Delta z}^2}} = \frac{\Delta x}{\Delta s} \cdot \frac{\Delta s}{\sqrt{\overline{\Delta x}^2+\overline{\Delta y}^2+\overline{\Delta z}^2}},$$

whence
$$\operatorname*{Lim}_{\Delta s \doteq 0} \frac{\Delta x}{\sqrt{\overline{\Delta x}^2+\overline{\Delta y}^2+\overline{\Delta z}^2}} = \frac{dx}{ds},$$

for
$$\operatorname{Lim} \frac{\Delta s}{\sqrt{\overline{\Delta x}^2+\overline{\Delta y}^2+\overline{\Delta z}^2}} = 1. \qquad (\text{I}, \S\ 104)$$

Proceeding in the same way with the other two ratios, we have $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ as the direction cosines of the curve at any point, since the directions of the tangent and the curve at any point are the same.

Instead of giving the direction cosines, we may speak of the direction of the curve as the direction $dx : dy : dz$, since

$$ds = \sqrt{dx^2 + dy^2 + dz^2}. \qquad (\S\ 94)$$

Ex. Find the direction of the helix

$$x = a \cos\theta, \quad y = a \sin\theta, \quad z = k\theta,$$

at the point for which $\theta = 0$.

Here $dx = -a \sin\theta\, d\theta$, $dy = a \cos\theta\, d\theta$, $dz = k\, d\theta$. Therefore, at the point for which $\theta = 0$, the direction is the direction $0 : a\, d\theta : k\, d\theta$, and the direction cosines are $0, \frac{a}{\sqrt{a^2+k^2}}, \frac{k}{\sqrt{a^2+k^2}}$.

98. Angle between two straight lines. Let the directions of any two straight lines be respectively $\Delta_1 x : \Delta_1 y : \Delta_1 z$ and $\Delta_2 x : \Delta_2 y : \Delta_2 z$, where the subscripts are used merely to distinguish the two directions. If two straight lines having these directions are drawn from any point $P(x, y, z)$ they will pass through the two points $P_1(x + \Delta_1 x, y + \Delta_1 y, z + \Delta_1 z)$ and $P_2(x + \Delta_2 x, y + \Delta_2 y, z + \Delta_2 z)$ (fig. 62) respectively.

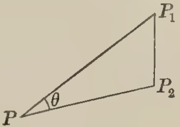

FIG. 62

Then if θ is the angle between these two lines, we have, by trigonometry,

$$\cos\theta = \frac{\overline{PP_1}^2 + \overline{PP_2}^2 - \overline{P_1P_2}^2}{2\,PP_1 \cdot PP_2}. \tag{1}$$

But
$$\overline{PP_1}^2 = \overline{\Delta_1 x}^2 + \overline{\Delta_1 y}^2 + \overline{\Delta_1 z}^2,$$
$$\overline{PP_2}^2 = \overline{\Delta_2 x}^2 + \overline{\Delta_2 y}^2 + \overline{\Delta_2 z}^2,$$
$$\overline{P_1P_2}^2 = (\Delta_2 x - \Delta_1 x)^2 + (\Delta_2 y - \Delta_1 y)^2 + (\Delta_2 z - \Delta_1 z)^2, \qquad (\S\ 93)$$

whence, by substitution in (1) and simplification,

$$\cos\theta = \frac{\Delta_1 x \cdot \Delta_2 x + \Delta_1 y \cdot \Delta_2 y + \Delta_1 z \cdot \Delta_2 z}{\sqrt{\overline{\Delta_1 x}^2 + \overline{\Delta_1 y}^2 + \overline{\Delta_1 z}^2} \cdot \sqrt{\overline{\Delta_2 x}^2 + \overline{\Delta_2 y}^2 + \overline{\Delta_2 z}^2}} \tag{2}$$

or if $\cos\alpha_1$, $\cos\beta_1$, $\cos\gamma_1$ are the direction cosines of any straight line with the first direction, and $\cos\alpha_2$, $\cos\beta_2$, $\cos\gamma_2$ are the direction cosines of any straight line with the second direction, the above formula becomes

$$\cos\theta = \cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2. \tag{3}$$

If the two directions are given as $A_1 : B_1 : C_1$, and $A_2 : B_2 : C_2$, (2) evidently becomes

$$\cos\theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2} \cdot \sqrt{A_2^2 + B_2^2 + C_2^2}}. \tag{4}$$

If the lines are perpendicular to each other,

$$A_1A_2 + B_1B_2 + C_1C_2 = 0, \tag{5}$$

since $\cos\theta = 0$; and if they are parallel to each other,

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}, \tag{6}$$

since $\quad \cos\alpha_1 = \cos\alpha_2, \quad \cos\beta_1 = \cos\beta_2, \quad \cos\gamma_1 = \cos\gamma_2.$

If θ represents the angle between any two curves, we have, from § 97,

$$\cos\theta = \left(\frac{dx}{ds}\right)_1\left(\frac{dx}{ds}\right)_2 + \left(\frac{dy}{ds}\right)_1\left(\frac{dy}{ds}\right)_2 + \left(\frac{dz}{ds}\right)_1\left(\frac{dz}{ds}\right)_2. \tag{7}$$

99. Direction of the normal to a plane. Let $P_1(x_1, y_1, z_1)$ and $P_2(x_1+\Delta x, y_1+\Delta y, z_1+\Delta z)$ be any two points of the plane

$$Ax + By + Cz + D = 0. \tag{1}$$

Substituting their coördinates in (1), we have

$$Ax_1 + By_1 + Cz_1 + D = 0, \tag{2}$$

$$A(x_1+\Delta x) + B(y_1+\Delta y) + C(z_1+\Delta z) + D = 0. \tag{3}$$

Subtracting (2) from (3), we have

$$A\cdot\Delta x + B\cdot\Delta y + C\cdot\Delta z = 0, \tag{4}$$

whence, by (5), § 98, the direction $A:B:C$ is normal to the direction $\Delta x:\Delta y:\Delta z$. But the latter direction is the direction of any straight line of the plane. Hence *the direction $A:B:C$ is the direction of the normal to the plane $Ax+By+Cz+D=0$.*

We may now show that the equation of any plane is a linear equation of the form (1). For let the given plane pass through a fixed point $P_1(x_1, y_1, z_1)$ and be perpendicular to a straight line having the direction $A:B:C$.

Now the equation $Ax + By + Cz + D = 0$

represents a plane perpendicular to the direction $A:B:C$. This plane will pass through P_1 if

$$Ax_1 + By_1 + Cz_1 + D = 0,$$

whence $$D = -(Ax_1 + By_1 + Cz_1).$$

Therefore $$A(x-x_1) + B(y-y_1) + C(z-z_1) = 0 \tag{5}$$

represents a plane perpendicular to the direction $A:B:C$ and passing through the fixed point $P_1(x_1, y_1, z_1)$. But only one plane can satisfy these conditions; hence the given plane has the equation just determined. Any plane may be determined in this way, however, and hence every plane may be represented by a linear equation.

Ex. Find the equation of a plane passing through the point (1, 2, 1) and normal to the straight line having the direction $2:3:-1$.

The equation is
$$2(x-1)+3(y-2)-1(z-1)=0,$$
or
$$2x+3y-z-7=0.$$

100. Normal equation of a plane. Let a plane be passed through the point $P_1(x_1, y_1, z_1)$ perpendicular to the straight line OP_1 (fig. 63). By § 96, the direction of OP_1 is the direction $x_1:y_1:z_1$, and hence the equation of the required plane is, by (5), § 99,

$$x_1(x-x_1)+y_1(y-y_1)+z_1(z-z_1)=0, \qquad (1)$$

which may be put in the form

$$x_1x+y_1y+z_1z-(x_1^2+y_1^2+z_1^2)=0. \quad (2)$$

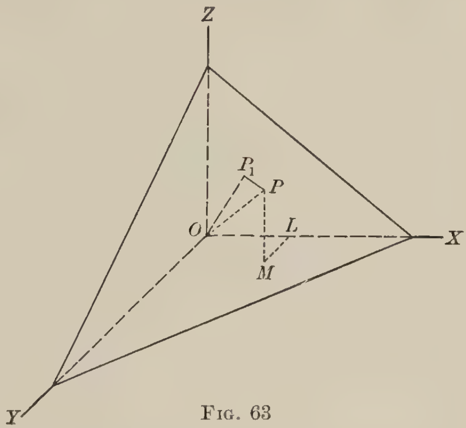

FIG. 63

If the direction cosines of OP_1 are denoted respectively by l, m, and n, i.e. $l=\cos\alpha$, $m=\cos\beta$, $n=\cos\gamma$, and the length of OP_1 is denoted by p, then

$$l=\frac{x_1}{\sqrt{x_1^2+y_1^2+z_1^2}}, \quad m=\frac{y_1}{\sqrt{x_1^2+y_1^2+z_1^2}}, \quad n=\frac{z_1}{\sqrt{x_1^2+y_1^2+z_1^2}},$$

and
$$p=\sqrt{x_1^2+y_1^2+z_1^2}.$$

Accordingly, if we divide equation (2) by $\sqrt{x_1^2+y_1^2+z_1^2}$, it becomes

$$lx+my+nz-p=0, \qquad (3)$$

which is known as the *normal equation* of the plane.

Equation (3) may also be derived geometrically as follows: Let $P(x, y, z)$ (fig. 63) be any point of the plane, and let $OL=x$, $LM=y$, $MP=z$. Draw OP. Then the projection of OP on OP_1 is OP_1, and the projection of the broken line $OLMP$ on OP_1 is $l\cdot OL+m\cdot LM+n\cdot MP$ (§ 92). Hence

$$l\cdot OL+m\cdot LM+n\cdot MP-OP_1=0,$$
or
$$lx+my+nz-p=0.$$

The general equation of the plane

$$Ax + By + Cz + D = 0$$

may evidently be made to assume the normal form by dividing the equation by $\sqrt{A^2 + B^2 + C^2}$, since the direction of the normal to the plane is the direction $A : B : C$. The sign of the radical must be taken opposite to the sign of D in order that the constant term may be the negative of the distance of the plane from the origin, as in (3).

Ex. Find the direction cosines of the normal to the plane $2x - 3y + 6z + 14 = 0$, also its distance from the origin.

Dividing by $-\sqrt{2^2 + 3^2 + 6^2}$, i.e. -7,

we have $$-\tfrac{2}{7}x + \tfrac{3}{7}y - \tfrac{6}{7}z - 2 = 0.$$

Hence the direction cosines of the normal to this plane are $-\frac{2}{7}$, $\frac{3}{7}$, $-\frac{6}{7}$, and the plane is 2 units distant from the origin.

101. Angle between two planes. Let the two planes be

$$A_1x + B_1y + C_1z + D_1 = 0, \tag{1}$$

$$A_2x + B_2y + C_2z + D_2 = 0. \tag{2}$$

The angle between these planes is the same as the angle between their respective normals, the directions of which are respectively the directions $A_1 : B_1 : C_1$ and $A_2 : B_2 : C_2$. Hence if θ is the angle between the two planes,

$$\cos\theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2} \cdot \sqrt{A_2^2 + B_2^2 + C_2^2}} \cdot \quad \text{(by (4), § 98)}$$

The conditions for perpendicularity and parallelism of the planes are respectively

$$A_1A_2 + B_1B_2 + C_1C_2 = 0$$

and

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}.$$

102. Determination of the direction cosines of any straight line. Let the equations of the straight line be in the form

$$A_1x + B_1y + C_1z + D_1 = 0, \tag{1}$$

$$A_2x + B_2y + C_2z + D_2 = 0, \tag{2}$$

and let its direction cosines be l, m, and n. Since the line lies in both planes (1) and (2), it is perpendicular to the normal to each. Therefore (by (5), § 98),

$$A_1 l + B_1 m + C_1 n = 0,$$

$$A_2 l + B_2 m + C_2 n = 0;$$

also $$l^2 + m^2 + n^2 = 1. \qquad (§ 95)$$

Here are three equations from which the values of l, m, and n may be found.

From the first two equations we have (I, § 8)

$$l : m : n = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} : \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix} : \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}.$$

Ex. Find the direction cosines of the straight line $2x + 3y + z - 4 = 0$, $4x + y - z + 7 = 0$.

The three equations for l, m, and n are

$$2l + 3m + n = 0,$$

$$4l + m - n = 0,$$

$$l^2 + m^2 + n^2 = 1,$$

the solutions of which are

$$l = \frac{2}{\sqrt{38}}, \quad m = -\frac{3}{\sqrt{38}}, \quad n = \frac{5}{\sqrt{38}},$$

or $$l = -\frac{2}{\sqrt{38}}, \quad m = \frac{3}{\sqrt{38}}, \quad n = -\frac{5}{\sqrt{38}}.$$

Since $\cos(180° - \phi) = -\cos\phi$, it is evident that if the angles corresponding to the first solution are α_1, β_1, γ_1, the angles corresponding to the second solution are $180° - \alpha_1$, $180° - \beta_1$, $180° - \gamma_1$. Since these two directions are each the negative of the other, it is sufficient to take either solution and ignore the other.

103. Equations of a straight line determined by two points. Let $P(x, y, z)$ be a point on the straight line determined by $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, so situated that $P_1P = k(P_1P_2)$. Then, as in I, § 18, there are three cases to consider according to the position of the point P. If P is between P_1 and P_2, P_1P and P_1P_2 have the same direction, and $P_1P < P_1P_2$; accordingly k is a positive number less than unity. If P is beyond P_2 from P_1, P_1P and P_1P_2 still have the same direction, but $P_1P > P_1P_2$; therefore k is a positive number greater than unity. Finally, if P

is beyond P_1 from P_2, P_1P and P_1P_2 have opposite directions, and k is a negative number, its numerical value ranging all the way from 0 to ∞.

In fig. 64, which represents the first case, pass planes through P_1, P, and P_2 perpendicular to OX; and let them intersect OX at the points M_1, M, and M_2 respectively. Then $OM = OM_1 + M_1M$; and since $P_1P = k(P_1P_2)$, by geometry $M_1M = k(M_1M_2)$.

$$\therefore OM = OM_1 + k(M_1M_2),$$

whence, by substitution,

$$x = x_1 + k(x_2 - x_1). \tag{1}$$

By passing the planes through the points perpendicular to OY and perpendicular to OZ, it may be proved in the same way that

$$y = y_1 + k(y_2 - y_1), \tag{2}$$

$$z = z_1 + k(z_2 - z_1). \tag{3}$$

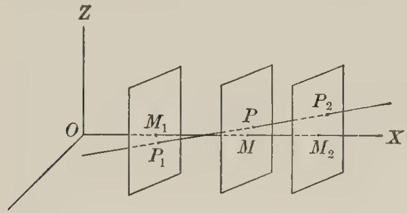

FIG. 64

The construction and the proof for the other two cases are the same, and the results are the same in all the cases.

By assigning different values to k we can make equations (1), (2), and (3) represent any point on the straight line determined by the points P_1 and P_2. Conversely, if x, y, and z satisfy these three equations, the point must be a point of the straight line P_1P_2. Hence (1), (2), and (3) are the parametric equations of the straight line determined by two points, k being the variable parameter.

104. By eliminating k from the three equations (1), (2), and (3) of the last article, we have

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \tag{1}$$

Here are but two independent equations in x, y, and z. This result proves the converse of § 89, that two linear equations always represent a straight line; for we have any straight line represented by two linear equations. The direction of the line is the direction $x_2 - x_1 : y_2 - y_1 : z_2 - z_1$ (§ 96).

It is to be noted that, if in the formation of these fractions any denominator is zero, the corresponding direction cosine is zero, and the line is perpendicular to the corresponding axis.

Ex. Find the equations of the straight line determined by the points $(1, 5, -1)$ and $(2, -3, -1)$.

$$\frac{x-1}{2-1} = \frac{y-5}{-3-5} = \frac{z+1}{-1+1}$$

Hence the two equations of the line are $z + 1 = 0$, since the line is parallel to the XOY plane and passes through a point for which $z = -1$, and $8x + y - 13 = 0$, formed by equating the first two fractions.

105. Equations of a tangent line to a space curve. If $P_1(x_1, y_1, z_1)$ is any given point of a space curve, and $P_2(x_1 + \Delta x, y_1 + \Delta y, z_1 + \Delta z)$ is any second point of the curve, the equations of the secant P_1P_2 are

$$\frac{x - x_1}{\Delta x} = \frac{y - y_1}{\Delta y} = \frac{z - z_1}{\Delta z}, \qquad (§\ 104)$$

or

$$\frac{x - x_1}{\dfrac{\Delta x}{\Delta s}} = \frac{y - y_1}{\dfrac{\Delta y}{\Delta s}} = \frac{z - z_1}{\dfrac{\Delta z}{\Delta s}},$$

where Δs is the length of the arc P_1P_2.

Defining the tangent at P_1 as in I, § 59, we have as its equations

$$\frac{x - x_1}{\left(\dfrac{dx}{ds}\right)_1} = \frac{y - y_1}{\left(\dfrac{dy}{ds}\right)_1} = \frac{z - z_1}{\left(\dfrac{dz}{ds}\right)_1},$$

or

$$\frac{x - x_1}{dx} = \frac{y - y_1}{dy} = \frac{z - z_1}{dz},$$

where dx, dy, dz are to be computed at the point P_1.

Ex. Find the equations of the tangent to the helix

$$x = a\cos\theta, \quad y = a\sin\theta, \quad z = k\theta$$

at the origin.

Since $dx = -a\sin\theta\, d\theta$, $dy = a\cos\theta\, d\theta$, $dz = k\, d\theta$, the required equations are $\frac{x}{0} = \frac{y}{a} = \frac{z}{k}$, or (§ 104) $x = 0$, $\frac{y}{a} = \frac{z}{k}$.

106. Equations of a straight line in terms of its direction cosines and a known point upon it. Let $P_1(x_1, y_1, z_1)$ (fig. 65) be a known point of the line, and let l, m, and n be its direction cosines. Let $P(x, y, z)$ be any point of the line. On P_1P as a diagonal construct a parallelopiped as in § 93. Then if we denote P_1P by r, we have

$$P_1Q = lr, \qquad P_1R = mr, \qquad P_1S = nr.$$

But $$P_1Q = x - x_1, \quad P_1R = y - y_1, \quad P_1S = z - z_1,$$

whence $$x = x_1 + lr, \qquad y = y_1 + mr, \qquad z = z_1 + nr. \tag{1}$$

These are the parametric equations of the line, the variable parameter being r, r being positive if the point is in one direction from P_1, and negative if it is in the other direction from P_1.

By eliminating r we have

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}, \tag{2}$$

which are but two independent linear equations.

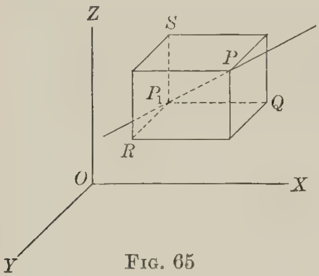

FIG. 65

107. Problems on the plane and the straight line. In this article we shall solve some problems illustrating the use of the equations of the plane and the straight line.

1. *Plane determined by three known points.* Let the three given points be $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, and $P_3(x_3, y_3; z_3)$, and let the equation of the plane determined by them be

$$Ax + By + Cz + D = 0. \tag{1}$$

Since P_1, P_2, and P_3 are points of the plane, their coördinates satisfy (1). Therefore

$$Ax_1 + By_1 + Cz_1 + D = 0, \tag{2}$$

$$Ax_2 + By_2 + Cz_2 + D = 0, \tag{3}$$

$$Ax_3 + By_3 + Cz_3 + D = 0. \tag{4}$$

We may now solve (2), (3), and (4) for the ratios of the unknown constants A, B, C, and D, and substitute in (1), or we may eliminate

A, B, C, and D from the four equations (1), (2), (3), and (4). By either method the equation of the required plane is found to be

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

Ex. 1. Find the equation of the plane determined by the three points $(1, 1, 1)$, $(-1, 1, 2)$, and $(2, -3, -1)$.

The required equation is

$$\begin{vmatrix} x & y & z & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 2 & 1 \\ 2 & -3 & -1 & 1 \end{vmatrix} = 0,$$

or $$4x - 3y + 8z - 9 = 0.$$

If the three given points are on the same straight line, no plane is determined, and the above equation reduces to $0 = 0$.

2. *Distance of a point from a plane.* Let the given point be $P_1(x_1, y_1, z_1)$, and at first let the equation of the plane be in the normal form

$$lx + my + nz - p = 0. \tag{1}$$

Then the equation of a plane through P_1 parallel to (1) is, by (5), § 99,

$$l(x - x_1) + m(y - y_1) + n(z - z_1) = 0, \tag{2}$$

which may be put in the form

$$lx + my + nz - (lx_1 + my_1 + nz_1) = 0. \tag{3}$$

Since equation (3) is in normal form, it is evident that the distance of this plane from the origin is $lx_1 + my_1 + nz_1$. Hence the distance between the two parallel planes is in magnitude

$$lx_1 + my_1 + nz_1 - p, \tag{4}$$

this result being positive if the plane through P_1 is beyond plane (1) from the origin, and negative if the plane through P_1 is on the same side of plane (1) as the origin. But the distance between the two planes is the same as the distance of P_1 from the given plane. Hence $lx_1 + my_1 + nz_1 - p$ is the required distance in magnitude.

If the equation of the plane is in the form

$$Ax + By + Cz + D = 0,$$

finding the values of l, m, n, and p, and substituting in (4), we have

$$\frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}$$

as the magnitude of the required distance, being positive for all points on one side of the plane and negative for all points on the other side. If we choose, we may take the sign of the radical always positive, in which case we can determine for which side of the plane the above result is positive by testing for some one point, preferably the origin.

Ex. 2. Find the distance of the point (1, 2, 1) from the plane $2x - 3y + 6z + 14 = 0$. The required distance is

$$\frac{2(1) - 3(2) + 6(1) + 14}{7} = 2\tfrac{2}{7}.$$

Furthermore the point is on the same side of the plane as the origin, for if (0, 0, 0) had been substituted, the result would have been 2, i.e. of same sign as $2\frac{2}{7}$.

3. *Plane through a given line and subject to one other condition.* Let the given line be

$$A_1x + B_1y + C_1z + D_1 = 0, \qquad (1)$$

$$A_2x + B_2y + C_2z + D_2 = 0. \qquad (2)$$

Multiplying the left-hand members of (1) and (2) by k_1 and k_2 respectively, where k_1 and k_2 are any two quantities independent of x, y, and z, and placing the sum of these products equal to zero, we have the equation

$$k_1(A_1x + B_1y + C_1z + D_1) + k_2(A_2x + B_2y + C_2z + D_2) = 0. \qquad (3)$$

Equation (3) is the equation of a plane, since it is a linear equation, and furthermore it passes through the given straight line, since the coördinates of every point of that line satisfy (3) by virtue of (1) and (2). Hence (3) is the required plane, and it may be made to satisfy another condition by determining the values of k_1 and k_2 appropriately.

Ex. 3. Find the equation of the plane determined by the point (0, 1, 0) and the line $4x + 3y + 2z - 4 = 0$, $2x - 11y - 4z - 12 = 0$.

The equation of the required plane may be written

$$k_1(4x + 3y + 2z - 4) + k_2(2x - 11y - 4z - 12) = 0. \quad (1)$$

Since (0, 1, 0) is a point of this plane, its coördinates satisfy (1), and hence

$$k_1 + 23k_2 = 0, \text{ or } k_1 = -23k_2.$$

Substituting this value of k_1 in (1) and reducing, we have as the required equation

$$9x + 8y + 5z - 8 = 0.$$

Ex. 4. Find the equation of the plane passing through the line $4x + 3y + 2z - 4 = 0$, $2x - 11y - 4z - 12 = 0$, and perpendicular to the plane $2x + y - 2z + 1 = 0$.

The equation of the required plane may be written

$$k_1(4x + 3y + 2z - 4) + k_2(2x - 11y - 4z - 12) = 0, \quad (1)$$

or $$(4k_1 + 2k_2)x + (3k_1 - 11k_2)y + (2k_1 - 4k_2)z + (-4k_1 - 12k_2) = 0.$$

Since this plane is to be perpendicular to the plane $2x + y - 2z + 1 = 0$,

$$2(4k_1 + 2k_2) + 1(3k_1 - 11k_2) - 2(2k_1 - 4k_2) = 0,$$

whence $k_2 = -7k_1$.

Substituting this value of k_2 in (1) and reducing, we have as the required equation

$$x - 8y - 3z - 8 = 0.$$

108. Change of coördinates. 1. *Change of origin without change of direction of axes.* Let $O'(x_0, y_0, z_0)$ be taken as a new origin of coördinates, the new axes being parallel respectively to the original axes, i.e. $O'X'$ parallel to OX, $O'Y'$ parallel to OY, and $O'Z'$ parallel to OZ. Let x, y, and z be the coördinates of any point P with respect to the original axes, and let x', y', and z' be the coördinates of the same point with respect to the new axes. Then

$$x = x_0 + x', \qquad y = y_0 + y', \qquad z = z_0 + z', \quad (1)$$

the proof being similar to that of I, § 112.

2. *Change of direction of axes without change of origin.* Let OX, OY, OZ, and OX', OY', OZ' be two sets of rectangular axes meeting at O and making angles with each other, whose cosines

are given in the following table, where l_1 is the cosine of the angle between OX and OX', l_2 is the cosine of the angle between OX and OY', etc. Let x, y, and z be the coördinates of any point P with respect to the axes OX, OY, OZ, and let x', y', and z' be the coördinates of the same point with respect to the axes OX', OY', OZ'. Then if OP is drawn, the projection of OP on OX is x, and the projection on OX of the broken line joining O and P, formed by constructing the coördinates x', y', z', is $l_1x' + l_2y' + l_3z'$ (§ 92). But these two projections are equal. Hence

	x'	y'	z'
x	l_1	l_2	l_3
y	m_1	m_2	m_3
z	n_1	n_2	n_3

$$x = l_1x' + l_2y' + l_3z';$$

in like manner,

$$y = m_1x' + m_2y' + m_3z', \tag{2}$$

$$z = n_1x' + n_2y' + n_3z'.$$

In like manner we may derive the formulas

$$\begin{aligned} x' &= l_1x + m_1y + n_1z, \\ y' &= l_2x + m_2y + n_2z, \\ z' &= l_3x + m_3y + n_3z. \end{aligned} \tag{3}$$

Formulas (2) and (3) may be expressed concisely by the above table.

Since both systems of coördinates are rectangular, we have, by (5), § 98,

$$\begin{aligned} l_1l_2 + m_1m_2 + n_1n_2 &= 0, \\ l_2l_3 + m_2m_3 + n_2n_3 &= 0, \\ l_3l_1 + m_3m_1 + n_3n_1 &= 0; \end{aligned}$$

and

$$\begin{aligned} l_1m_1 + l_2m_2 + l_3m_3 &= 0, \\ m_1n_1 + m_2n_2 + m_3n_3 &= 0, \\ n_1l_1 + n_2l_2 + n_3l_3 &= 0. \end{aligned}$$

Also, by § 95,

$$l_1^2+m_1^2+n_1^2=1,$$
$$l_2^2+m_2^2+n_2^2=1,$$
$$l_3^2+m_3^2+n_3^2=1;$$

and

$$l_1^2+l_2^2+l_3^2=1,$$
$$m_1^2+m_2^2+m_3^2=1,$$
$$n_1^2+n_2^2+n_3^2=1.$$

All these formulas are easily remembered by aid of the above table.

PROBLEMS

1. Find a point of the plane $2x+3y+2z=0$ equally distant from the three points $(0, 0, -1)$, $(3, 1, 1)$, $(-2, -1, 0)$.

2. Find a point on the line $3x-y-z-5=0$, $x-y+z-5=0$ equally distant from the origin and the point $(-2, 1, -2)$.

3. Find the equation of the sphere passing through the points $(-1, 4, 4)$, $(-5, 1, 3)$, $(4, 0, 7)$, $(-1, 1, -5)$.

4. A point moves so that its distances from two fixed points are in the constant ratio k. Prove that its locus is a sphere or a plane according as $k \neq 1$ or $k=1$.

5. Prove that the locus of points from which tangents of equal length can be drawn to two given spheres is a plane perpendicular to their line of centers.

6. Find the length of the curve $x=t^2$, $y=2t$, $z=t$ from the origin to the point for which $t=1$.

7. Find the length of the curve $x=e^t$, $y=e^{-t}$, $z=t\sqrt{2}$ between the points for which $t=0$ and $t=1$.

8. Find the length of the conical helix $x=t\cos t$, $y=t\sin t$, $z=t$ between the points for which $t=0$ and $t=2\pi$.

9. Prove that a straight line can make the angles 60°, 45°, 60° respectively with the coördinate axes.

10. Show that the helix makes a constant angle with the elements of the cylinder on which it is drawn.

11. Find the angle between the conical helix $x=t\cos t$, $y=t\sin t$, $z=t$ and the axis of the cone.

12. Show that the angle between the conical helix $x=t\cos t$, $y=t\sin t$, $z=t$ and the element of the cone is $\tan^{-1}\frac{t}{\sqrt{2}}$.

13. Find the equation of a plane three units distant from the origin and perpendicular to the straight line through the origin and $(2, -3, 6)$.

14. The equations of three planes are $x + 2y - 3z = 1$, $2x - 3y + 5z = 3$, and $7x - y - z = 2$. Find the equation of the plane through their point of intersection and equally inclined to the coördinate axes.

15. If the normal distance from the origin to the plane which makes intercepts a, b, and c respectively on the axes of x, y, and z is p, prove that $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$.

16. Find the equation of the plane normal to the helix $x = a\cos\theta$, $y = a\sin\theta$, $z = k\theta$ at the point $\theta = 0$.

17. Find the equation of the plane normal to the conical helix $x = t\cos t$, $y = t\sin t$, $z = t$ at the point for which $t = \frac{\pi}{2}$.

18. Find the equation of the normal plane to the curve $x = e^t$, $y = e^{-t}$, $z = t\sqrt{2}$.

19. Find the equation of the normal plane to the skew cubic $x = t$, $y = t^2$, $z = t^3$ at the point for which $t = 1$.

20. Find the direction cosines of the line $2x + 3y - z = 6$, $x + 3z - 7 = 0$.

21. Prove that the three planes $3x - 2y - 1 = 0$, $4y - 3z + 2 = 0$, $z - 2x + 4 = 0$ are the lateral faces of a triangular prism.

22. Prove that the two lines

$$\left\{\begin{array}{r} 3x + y - 7z + 11 = 0 \\ 2x + 4y + z - 3 = 0 \end{array}\right\} \text{ and } \left\{\begin{array}{r} x + 17y + 26z - 48 = 0 \\ 4x - 2y - 15z + 25 = 0 \end{array}\right\}$$

are coincident.

23. Prove that the lines

$$\left\{\begin{array}{r} y + 1 = 0 \\ 2x + 3y + 2z - 1 = 0 \end{array}\right\} \text{ and } \left\{\begin{array}{r} 2y - 3z + 5 = 0 \\ 7x - 4y - z - 10 = 0 \end{array}\right\}$$

intersect at right angles.

24. If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two given straight lines, and l, m, n are the direction cosines of a normal to each of them, prove that $\frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1}$.

25. Find the equation of the plane passing through (1, 4, 1) perpendicular to the line $x + 3y + 5z + 6 = 0$, $y + 2x - 1 = 0$.

26. Find the angle between the line $x - 2y - 8 = 0$, $3y + z + 8 = 0$ and the plane $3x + y - 2z + 7 = 0$.

27. Find the coördinates of a point on the straight line determined by (0, 1, 0) and (2, 3, 2) and 1 unit distant from (1, 1, 1).

28. Find the equation of a plane perpendicular to the straight line joining (1, 3, 5) and (4, 3, 2) at a point one third of the distance from the first to the second point.

29. Find the equations of a straight line passing through (1, 2, − 1) parallel to the line $x + 2y - 3z + 1 = 0$, $2x + y + 5z - 1 = 0$.

30. Find the foot of the perpendicular drawn from (1, 2, 1) to the plane $x - 3y + z - 7 = 0$.

31. Find the equations of the tangent to the skew quartic $x = t$, $y = t^3$, $z = t^4$ at the point for which $t = 1$.

32. Find the equations of the tangent to the curve $x = t^2$, $y = 2t$, $z = t$ at the point for which $t = 1$.

33. Find the equations of the tangent to the curve $x = e^t$, $y = e^{-t}$, $z = t\sqrt{2}$.

34. Find the direction of the conical helix $x = t\cos t$, $y = t\sin t$, $z = t$ at the origin, and the equations of the tangent.

35. Find the equation of the plane determined by the three points (1, 2, – 4), (3, – 1, 2), (2, 1, – 2).

36. Find the angles made with the coördinate planes by the plane determined by the three points (1, 2, 0), (4, 1, – 2), (– 2, 2, 2).

37. Find the point of intersection of the lines

$$\left\{\begin{array}{r}2x - y - 3 = 0\\ 3y - 2z + 5 = 0\end{array}\right\} \text{ and } \left\{\begin{array}{r}3x - 2y - 5 = 0\\ 2x - z - 1 = 0\end{array}\right\}.$$

38. Find whether or not a plane can be determined by the lines

$$\left\{\begin{array}{l}z = 4y - 7\\ z = 7 - 2x\end{array}\right\} \text{ and } \left\{\begin{array}{r}x - 2y + z + 9 = 0\\ 3x + 2y + z - 15 = 0\end{array}\right\}.$$

39. Find the equation of the plane determined by the two lines

$$\left\{\begin{array}{l}x + 2y + 1 = 0\\ 2y + z + 1 = 0\end{array}\right\} \text{ and } \left\{\begin{array}{l}7x + z - 24 = 0\\ 7y + 3z + 5 = 0\end{array}\right\}.$$

40. Find the equation of the plane determined by the point (2, 4, 2) and the straight line passing through the point (1, 2, 3) equally inclined to the coördinate axes.

41. Find the equation of a plane passing through the line $x - y + z = 0$, $2x + y + 3z = 0$ and perpendicular to the plane $x - y + 2z + 1 = 0$.

42. Find the equations of the projection of the line $x + y + z - 2 = 0$, $x + 2y + z - 2 = 0$ upon the plane $3x + y + 3z - 1 = 0$.

43. Find the equation of the plane passing through (2, – 1, 2) and (– 1, 2, – 1) perpendicular to the plane $2x - 3y + 2z - 6 = 0$.

44. Find the equation of a plane passing through the line $x - 2y + z - 3 = 0$, $2x + 3y - 2z - 1 = 0$ and parallel to the line $3x + y + 2z - 4 = 0$, $2x - 3y - z + 5 = 0$.

45. Prove that the plane $4x + 3y + 5z = 47$ is tangent to the sphere

$$(x - 2)^2 + (y - 3)^2 + (z + 4)^2 = 50.$$

46. Find a point on the line $x - z + 3 = 0$, $4x - y - 6 = 0$ equally distant from the planes $3x + 3z - 5 = 0$ and $x + 4y + z = 7$.

47. Find the center of a sphere of radius 7, passing through the points (2, 4, – 4) and (3, – 1, – 4) and tangent to the plane $3x - 6y + 2z + 51 = 0$.

CHAPTER XI

PARTIAL DIFFERENTIATION

109. Partial derivatives. Consider $f(x, y)$, where x and y are independent variables. We may, if we choose, allow x alone to vary, holding y temporarily constant. We thus reduce $f(x, y)$ to a function of x alone, which may have a derivative, defined and computed as for any function of one variable. This derivative is called the *partial derivative of* $f(x, y)$ *with respect to* x, and is denoted by the symbol $\frac{\partial f(x, y)}{\partial x}\cdot$ Thus, by definition,

$$\frac{\partial f(x, y)}{\partial x} = \lim_{\Delta x \doteq 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}. \tag{1}$$

Similarly, if x is held constant, $f(x, y)$ becomes temporarily a function of y, whose derivative is called the *partial derivative of* $f(x, y)$ *with respect to* y, denoted by the symbol $\frac{\partial f(x, y)}{\partial y}\cdot$ Then

$$\frac{\partial f(x, y)}{\partial y} = \lim_{\Delta y \doteq 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}. \tag{2}$$

Graphically, if $z = f(x, y)$ is represented by a surface, the relation between z and x when y is held constant is represented by the curve of intersection of the surface and the plane $y = \text{const.}$, and $\frac{\partial z}{\partial x}$ is the slope of this curve. Also, the relation between z and y when x is constant is represented by the curve of intersection of the surface and a plane $x = \text{const.}$, and $\frac{\partial z}{\partial y}$ is the slope of this curve.

Thus, in fig. 66, if $PQSR$ represents a portion of the surface $z = f(x, y)$, PQ is the curve $y = \text{const.}$, and PR is the curve $x = \text{const.}$ Let P be the point (x, y, z), and $LK = PK' = \Delta x$, $LM = PM' = \Delta y$.

Then $LP = f(x, y)$, $KQ = f(x + \Delta x, y)$, $MR = f(x, y + \Delta y)$, $K'Q = f(x + \Delta x, y) - f(x, y)$, $M'R = f(x, y + \Delta y) - f(x, y)$, and

$$\frac{\partial z}{\partial x} = \text{Lim}\,\frac{K'Q}{PK'} = \text{slope of } PQ,$$

$$\frac{\partial z}{\partial y} = \text{Lim}\,\frac{M'R}{PM'} = \text{slope of } PR.$$

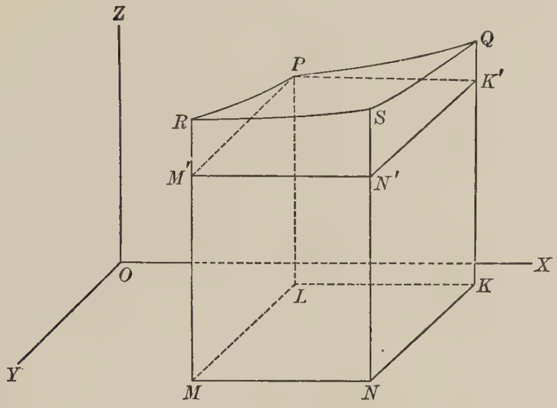

FIG. 66

Ex. 1. Consider a perfect gas obeying the law $v = \frac{ct}{p}$. We may change the temperature while keeping the pressure unchanged. The relation between the volume and the temperature is then represented by a straight line on fig. 55. If Δt and Δv are corresponding increments of t and v, then

$$\Delta v = \frac{c(t + \Delta t)}{p} - \frac{ct}{p} = \frac{c\Delta t}{p}$$

and
$$\frac{\partial v}{\partial t} = \frac{c}{p}.$$

Or, we may change the pressure while keeping the temperature unchanged. The relation between the volume and the pressure is then represented by an hyperbola on the surface of fig. 55. If Δp and Δv are corresponding increments of p and v, then

$$\Delta v = \frac{ct}{p + \Delta p} - \frac{ct}{p} = -\frac{ct\Delta p}{p^2 + p\Delta p}$$

and
$$\frac{\partial v}{\partial p} = -\frac{ct}{p^2}.$$

So, in general, if we have a function of any number of variables $f(x, y, \cdots, z)$, we may have a partial derivative with respect to each of the variables. These derivatives are expressed by the symbols $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \cdots, \frac{\partial f}{\partial z}$, or sometimes by $f_x(x, y, \cdots, z), f_y(x, y, \cdots, z), \cdots, f_z(x, y, \cdots, z)$. To compute these derivatives, we have to apply the formulas for the derivative of a function of one variable, regarding as constant all the variables except the one with respect to which we differentiate.

Ex. 2. $f = x^3 - 3x^2y + y^3$,

$$\frac{\partial f}{\partial x} = 3x^2 - 6xy,$$

$$\frac{\partial f}{\partial y} = -3x^2 + 3y^2.$$

Ex. 3. $f = \sin(x^2 + y^2)$,

$$\frac{\partial f}{\partial x} = 2x\cos(x^2 + y^2),$$

$$\frac{\partial f}{\partial y} = 2y\cos(x^2 + y^2).$$

Ex. 4. $f = \log\sqrt{x^2 + y^2 + z^2}$,

$$\frac{\partial f}{\partial x} = \frac{x}{x^2 + y^2 + z^2},$$

$$\frac{\partial f}{\partial y} = \frac{y}{x^2 + y^2 + z^2},$$

$$\frac{\partial f}{\partial z} = \frac{z}{x^2 + y^2 + z^2}.$$

110. Increment and total differential of a function of several variables. Consider $f(x, y)$, and let x and y be given any increments Δx and Δy. Then f takes an increment Δf, where

$$\Delta f = f(x + \Delta x, y + \Delta y) - f(x, y).$$

In fig. 66, $NS = f(x + \Delta x, y + \Delta y)$ and $N'S = \Delta f$. If x and y are independent variables, Δx and Δy are also independent. Thus the position of S in fig. 66 depends upon the choice of LK and LM, which can be taken at pleasure.

The function $f(x, y)$ is called a continuous function of x and y if Δf approaches zero as a limit when Δx and Δy approach zero as a limit in any manner whatever.

Thus in fig. 66, if z is a continuous function of x and y, the point S will approach the point P as LK and LM approach zero, no matter what curve the point N traces on the plane XOY or the point S on the surface.

The expression for Δf may be modified as follows:

$$\begin{aligned}\Delta f &= f(x+\Delta x,\, y+\Delta y)-f(x,\, y)\\ &= f(x+\Delta x,\, y+\Delta y)-f(x,\, y+\Delta y)+f(x,\, y+\Delta y)-f(x,\, y).\end{aligned}$$

But $\displaystyle\lim_{\Delta x \doteq 0}\frac{f(x+\Delta x,\, y+\Delta y)-f(x,\, y+\Delta y)}{\Delta x}=\frac{\partial f(x,\, y+\Delta y)}{\partial x}.$

Therefore $\displaystyle\frac{f(x+\Delta x,\, y+\Delta y)-f(x,\, y+\Delta y)}{\Delta x}=\frac{\partial f(x,\, y+\Delta y)}{\partial x}+\epsilon',$

or $\displaystyle f(x+\Delta x,\, y+\Delta y)-f(x,\, y+\Delta y)=\left(\frac{\partial f(x,\, y+\Delta y)}{\partial x}+\epsilon'\right)\Delta x,$

where $\displaystyle\lim_{\Delta x \doteq 0}\epsilon' = 0$. Also, since $\dfrac{\partial f}{\partial x}$ is a continuous function, $\dfrac{\partial f(x,\, y+\Delta y)}{\partial x}=\dfrac{\partial f(x,\, y)}{\partial x}+\epsilon''$, where $\displaystyle\lim_{\Delta y \doteq 0}\epsilon''=0$. Therefore

$$f(x+\Delta x,\, y+\Delta y)-f(x,\, y+\Delta y)=\left(\frac{\partial f(x,\, y)}{\partial x}+\epsilon_1\right)\Delta x,$$

where $\epsilon_1=\epsilon'+\epsilon''$.

Similarly, $\displaystyle f(x,\, y+\Delta y)-f(x,\, y)=\left(\frac{\partial f(x,\, y)}{\partial y}+\epsilon_2\right)\Delta y,$

where $\displaystyle\lim_{\Delta y \doteq 0}\epsilon_2=0$. Hence we have finally

$$\Delta f=\frac{\partial f}{\partial x}\Delta x+\frac{\partial f}{\partial y}\Delta y+\epsilon_1\Delta x+\epsilon_2\Delta y. \qquad (1)$$

In like manner, if f is a function of any number of variables $x, y, \cdots, z$, then

$$\Delta f=\frac{\partial f}{\partial x}\Delta x+\frac{\partial f}{\partial y}\Delta y+\cdots+\frac{\partial f}{\partial z}\Delta z+\epsilon_1\Delta x+\epsilon_2\Delta y+\cdots+\epsilon_n\Delta z. \quad (2)$$

In a manner analogous to the procedure in the case of a function of one variable (§ 4), we separate from the increment the terms $\epsilon_1\Delta x+\epsilon_2\Delta y+\cdots+\epsilon_n\Delta z$, call the remaining terms the *differential of the function*, and denote them by df. The differentials of the independent variables are taken equal to the increments, as in § 4. Thus, we have by definition, when f is a function of two independent variables x and y,

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy, \qquad (3)$$

and if f is a function of the independent variables $x, y, \cdots, z$,

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \cdots + \frac{\partial f}{\partial z}dz. \tag{4}$$

In (3) and (4) dx, dy, etc., may be given any values whatever. If, in particular, we place all but one equal to zero, we have the partial differentials, indicated by d_xf, d_yf, etc. Thus,

$$d_xf = \frac{\partial f}{\partial x}dx, \qquad d_yf = \frac{\partial f}{\partial y}dy.$$

A partial differential expresses approximately the change in the function caused by a change in one of the independent variables; the total differential expresses approximately the change in the function caused by changes in all the independent variables. It appears from (4) that the total differential is the sum of the partial differentials.

Ex. The period of a simple pendulum with small oscillations is (Ex. 3, § 82)

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

Small errors dl and dg, in determining l and g, will make an error in T of

$$dT = \frac{\partial T}{\partial l}dl + \frac{\partial T}{\partial g}dg = \frac{\pi}{\sqrt{lg}}dl - \frac{\pi}{g}\sqrt{\frac{l}{g}}dg.$$

The ratio of error is

$$\frac{dT}{T} = \frac{1}{2}\frac{dl}{l} - \frac{1}{2}\frac{dg}{g}.$$

111. Derivative of $f(x, y)$ when x and y are functions of t. We have been considering $f(x, y)$ as a function of two independent variables. We shall now suppose that x and y are functions of a single independent variable t, so that the variations of x and y are caused by the variations of t. Graphically, if $z = f(x, y)$ represents a surface and x and y are independent, the point $P(x, y)$ may move over the entire surface. If, however, $x = \phi_1(t)$, $y = \phi_2(t)$, the point P is restricted to a curve on the surface, whose projection on the plane XOY has the parametric equations $x = \phi_1(t)$, $y = \phi_2(t)$. The equations of the curve on the surface are, therefore, $x = \phi_1(t)$, $y = \phi_2(t)$, $z = f[\phi_1(t), \phi_2(t)]$.

In this way, $f(x, y)$ is now a function of t and may have a derivative with respect to t, which may be found as follows:

Give t an increment Δt. Then x and y take increments Δx and Δy, and f in turn receives an increment Δf, where

$$\Delta f = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y.$$

Then
$$\frac{\Delta f}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t} + \epsilon_1 \frac{\Delta x}{\Delta t} + \epsilon_2 \frac{\Delta y}{\Delta t}.$$

By allowing Δt to approach zero as a limit, and taking the limits, we have
$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}. \tag{1}$$

If we multiply each term of (1) by the differential dt, we have

$$\frac{df}{dt}dt = \frac{\partial f}{\partial x}\frac{dx}{dt}dt + \frac{\partial f}{\partial y}\frac{dy}{dt}dt.$$

But since f, x, and y are functions of a single variable t, we have, by § 4,
$$df = \frac{df}{dt}dt, \quad dx = \frac{dx}{dt}dt, \quad dy = \frac{dy}{dt}dt.$$

Hence we have
$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy, \tag{2}$$

showing that formula (3) of § 110, made on the hypothesis that x and y were independent, holds also when x and y are functions of t.

In a similar manner, if we have $f(x, y, \cdots, z)$, where $x, y, \cdots, z$ are all functions of t, then

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \cdots + \frac{\partial f}{\partial z}\frac{dz}{dt}, \tag{3}$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \cdots + \frac{\partial f}{\partial z}dz. \tag{4}$$

Ex. Let $V(x, y, z)$ be the electrical potential at a point in an electrified field, that is, let $\frac{\partial V}{\partial x} = -X$, $\frac{\partial V}{\partial y} = -Y$, $\frac{\partial V}{\partial z} = -Z$, where X, Y, Z are the components of force in the directions of the coördinate axes. Required the rate of change of V in a direction which makes the angles α, β, γ with the axes.

A straight line making the given angles with the axes has the equations, by (1), § 106,

$$x = x_1 + r\cos\alpha,$$
$$y = y_1 + r\cos\beta,$$
$$z = z_1 + r\cos\gamma.$$

If these values are substituted for x, y, and z in V, it becomes a function of r, and its rate of change with respect to r is $\frac{dV}{dr}$. By (3),

$$\begin{aligned}\frac{dV}{dr} &= \frac{\partial V}{\partial x}\frac{dx}{dr} + \frac{\partial V}{\partial y}\frac{dy}{dr} + \frac{\partial V}{\partial z}\frac{dz}{dr}\\ &= -X\cos\alpha - Y\cos\beta - Z\cos\gamma.\end{aligned}$$

By the principle of the composition of forces this is minus the component of force in the given direction.

112. Tangent plane to a surface. Let $x = \phi_1(t)$, $y = \phi_2(t)$, $z = f(x, y)$ be a curve on the surface $z = f(x, y)$, and let $P_1(x_1, y_1, z_1)$ be a point on the curve. Then the tangent line to the curve at the point P_1 has the equations (§ 105)

$$\frac{x - x_1}{dx} = \frac{y - y_1}{dy} = \frac{z - z_1}{dz},$$

or

$$\frac{x - x_1}{dx} = \frac{y - y_1}{dy} = \frac{z - z_1}{\left(\frac{\partial f}{\partial x}\right)_1 dx + \left(\frac{\partial f}{\partial y}\right)_1 dy}. \qquad (1)$$

The line (1) is perpendicular to the straight line

$$\frac{x - x_1}{\left(\frac{\partial f}{\partial x}\right)_1} = \frac{y - y_1}{\left(\frac{\partial f}{\partial y}\right)_1} = \frac{z - z_1}{-1}, \qquad (2)$$

by (5), § 98, and therefore lies in the plane

$$(x - x_1)\left(\frac{\partial f}{\partial x}\right)_1 + (y - y_1)\left(\frac{\partial f}{\partial y}\right)_1 - (z - z_1) = 0. \qquad (3)$$

Equations (2) and (3) are independent of the functions which define the curve, and therefore the tangents to all curves through P_1 lie in the plane (3). This plane is called the *tangent plane* to the surface, and the line (2), which is normal to it, is called the *normal line* to the surface at the point P_1. The tangent plane may simply touch the surface, as in the case of the sphere or the ellipsoid, or it may intersect the surface at the point of tangency, as in the case of the hyperboloid of one sheet.

In order that the function $f(x, y)$ shall have a maximum or a minimum value for $x = x_1$, $y = y_1$, it is necessary, but not sufficient, that the tangent plane to the surface $z = f(x, y)$ at the point (x_1, y_1, z_1) should be parallel to the plane XOY. This occurs when $\left(\frac{\partial f}{\partial x}\right)_1 = 0$, $\left(\frac{\partial f}{\partial y}\right)_1 = 0$. These are therefore *necessary* conditions for a maximum or a minimum, and in case the existence of a maximum or a minimum is known from the nature of the problem, it may be located by solving these equations.

Ex. 1. Find the tangent plane and the normal line to the paraboloid $z = ax^2 + by^2$.

Here $\frac{\partial z}{\partial x} = 2\,ax$ and $\frac{\partial z}{\partial y} = 2\,by$. Hence the tangent plane is

$$(x - x_1)\,2\,ax_1 + (y - y_1)\,2\,by_1 - (z - z_1) = 0,$$

or

$$2\,ax_1x + 2\,by_1y - 2\,ax_1^2 - 2\,by_1^2 - z + z_1 = 0.$$

But since $2\,ax_1^2 + 2\,by_1^2 = 2\,z_1$, this may be written

$$2\,ax_1x + 2\,by_1y - z - z_1 = 0.$$

The normal is

$$\frac{x - x_1}{2\,ax_1} = \frac{y - y_1}{2\,by_1} = \frac{z - z_1}{-1}.$$

Ex. 2. It is required to construct out of a given amount of material a cistern in the form of a rectangular parallelopiped open at the top. Required the dimensions in order that the capacity may be a maximum, if no allowance is made for thickness of the material or waste in construction.

Let x, y, z be the length, the breadth, and the height respectively. Then the superficial area is $xy + 2\,xz + 2\,yz$, which may be placed equal to the given amount of material, a. If v is the capacity of the cistern,

$$v = xyz = \frac{axy - x^2y^2}{2\,(x + y)}.$$

Then

$$\frac{\partial v}{\partial x} = \frac{(a - 2\,xy - x^2)\,y^2}{2\,(x + y)^2}, \quad \frac{\partial v}{\partial y} = \frac{(a - 2\,xy - y^2)\,x^2}{2\,(x + y)^2}.$$

For the maximum these must be zero, and since it is not admissible to have $x = 0$, $y = 0$, we have to solve the equations

$$a - 2\,xy - x^2 = 0,$$
$$a - 2\,xy - y^2 = 0,$$

which have for the only positive solutions $x = y = \sqrt{\frac{a}{3}}$, whence $z = \frac{1}{2}\sqrt{\frac{a}{3}}$. Consequently, if there is a maximum capacity, it must be for these dimensions. It is very evident that a maximum does exist; hence the problem is solved.

The graphical interpretation of the differential can now be given in a manner analogous to that in the case of one variable (§ 5). In fig. 67 let $PQ'S'R'$ be the tangent plane at $P(x_1, y_1, z_1)$. Then, if $LK = dx$ and $LM = dy$, the coördinates of S are $(x_1 + dx, y_1 + dy, z_1 + \Delta z)$, and the value of z corresponding to S' is found by replacing x by $x_1 + dx$ and y by $y_1 + dy$ in (3). There results $z = z_1 + \left(\frac{\partial z}{\partial x}\right)_1 dx + \left(\frac{\partial z}{\partial y}\right)_1 dy = z_1 + dz$. Therefore

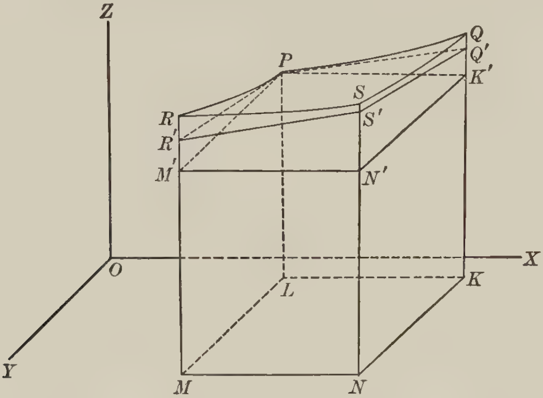

FIG. 67

$$N'S' = dz, \text{ whereas } N'S = \Delta z.$$

113. Derivatives of $f(x, y)$ when x and y are functions of s and t. If x and y are functions of two independent variables s and t, then $f(x, y)$ is also a function of s and t and may have the derivatives $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$. To find $\frac{\partial f}{\partial s}$ we will give s an increment Δs. Then x and y take increments Δx and Δy, and f takes an increment Δf. As in § 111, we find

$$\frac{\Delta f}{\Delta s} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta s} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta s} + \epsilon_1 \frac{\Delta x}{\Delta s} + \epsilon_2 \frac{\Delta y}{\Delta s}.$$

Now let Δs approach zero and take the limit, remembering that $\operatorname{Lim}\frac{\Delta f}{\Delta s}=\frac{\partial f}{\partial s}$, $\operatorname{Lim}\frac{\Delta x}{\Delta s}=\frac{\partial x}{\partial s}$, etc. We have

$$\frac{\partial f}{\partial s}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial s}.$$

Similarly,

$$\frac{\partial f}{\partial t}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial t}. \tag{1}$$

So, generally, the partial derivatives of $f(x, y, \cdots, z)$, where $x, y, \cdots, z$ are functions of $s, t, \cdots, u$, are

$$\begin{aligned}
\frac{\partial f}{\partial s}&=\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial s}+\cdots+\frac{\partial f}{\partial z}\frac{\partial z}{\partial s},\\
\frac{\partial f}{\partial t}&=\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial t}+\cdots+\frac{\partial f}{\partial z}\frac{\partial z}{\partial t},\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
\frac{\partial f}{\partial u}&=\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}+\cdots+\frac{\partial f}{\partial z}\frac{\partial z}{\partial u}.
\end{aligned} \tag{2}$$

A special case worth noting is that in which f is a function of x, where x is a function of $s, t, \cdots, u$. Then

$$\frac{\partial f}{\partial s}=\frac{df}{dx}\frac{\partial x}{\partial s},\quad \frac{\partial f}{\partial t}=\frac{df}{dx}\frac{\partial x}{\partial t},\quad \cdots,\quad \frac{\partial f}{\partial u}=\frac{df}{dx}\frac{\partial x}{\partial u}. \tag{3}$$

If we multiply the equations (2) by the differentials $ds, dt, \cdots, du$, add the results, and apply the definition of § 110, we have

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\cdots+\frac{\partial f}{\partial z}dz. \tag{4}$$

This shows that *the expression for the differential df is the same whether $x, y, \cdots, z$ are independent variables or not.*

Ex. 1. If $z=f(x-y, y-x)$, prove $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=0$.

Place $x-y=u$, $y-x=v$. Then $z=f(u, v)$, and

$$\begin{aligned}
\frac{\partial z}{\partial x}&=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial f}{\partial u}-\frac{\partial f}{\partial v},\\
\frac{\partial z}{\partial y}&=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}=-\frac{\partial f}{\partial u}+\frac{\partial f}{\partial v}.
\end{aligned}$$

By addition the required result is obtained.

Ex. 2. Let it be required to change from rectangular coördinates (x, y) to polar coördinates (r, θ), where $x = r\cos\theta$, $y = r\sin\theta$.

Then
$$\begin{aligned} \frac{\partial x}{\partial r} &= \cos\theta, & \frac{\partial y}{\partial r} &= \sin\theta, \\ \frac{\partial x}{\partial \theta} &= -r\sin\theta, & \frac{\partial y}{\partial \theta} &= r\cos\theta, \end{aligned} \tag{1}$$

and consequently, if f is a function of x and y,

$$\begin{aligned} \frac{\partial f}{\partial r} &= \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta, \\ \frac{\partial f}{\partial \theta} &= -r\frac{\partial f}{\partial x}\sin\theta + r\frac{\partial f}{\partial y}\cos\theta. \end{aligned} \tag{2}$$

Also, since $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$,

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{x}{\sqrt{x^2+y^2}} = \cos\theta, & \frac{\partial r}{\partial y} &= \frac{y}{\sqrt{x^2+y^2}} = \sin\theta, \\ \frac{\partial \theta}{\partial x} &= -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{r}, & \frac{\partial \theta}{\partial y} &= \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}; \end{aligned} \tag{3}$$

whence
$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial f}{\partial r}\cos\theta - \frac{\partial f}{\partial \theta}\frac{\sin\theta}{r}, \\ \frac{\partial f}{\partial y} &= \frac{\partial f}{\partial r}\sin\theta + \frac{\partial f}{\partial \theta}\frac{\cos\theta}{r}. \end{aligned} \tag{4}$$

Equations (4) may also be obtained from (2) by solving for $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

It is to be emphasized that $\frac{\partial x}{\partial r}$ in (1) is not the reciprocal of $\frac{\partial r}{\partial x}$ in (3). In (1) $\frac{\partial x}{\partial r}$ means the limit of the ratio of the increment of x to an increment of r when θ is constant. Graphically (fig. 68), $OP = r$ is increased by $PQ = \Delta r$, and $PR = \Delta x$ is thus determined. Then $\frac{\partial x}{\partial r} = \text{Lim}\,\frac{PR}{PQ} = \cos\theta$.

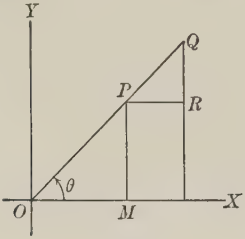

FIG. 68

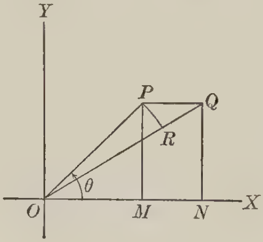

FIG. 69

Also $\frac{\partial r}{\partial x}$ in (3) means the limit of the ratio of the increment of r to that of x when y is constant. Graphically (fig. 69), $OM = x$ is increased by $MN = PQ = \Delta x$, and $RQ = \Delta r$ is thus determined. Then $\frac{\partial r}{\partial x} = \operatorname{Lim} \frac{RQ}{PQ} = \cos\theta$. It happens here that $\frac{\partial x}{\partial r} = \frac{\partial r}{\partial x}$. But $\frac{\partial x}{\partial \theta}$ in (1) and $\frac{\partial \theta}{\partial x}$ in (3) are neither equal nor reciprocal.

In cases where ambiguity is likely to arise as to which variable is constant in a partial derivative, the symbol for the derivative is sometimes inclosed in a parenthesis and the constant variable is written as a subscript, thus $\left(\frac{\partial x}{\partial r}\right)_\theta$.

Ex. 3. Consider $f(x, y, z)$ when $z = \phi(x, y)$. We may find $\frac{\partial f}{\partial x}$ from the first of formulas (2) in that we place $s = x$ and $t = y$. Then $\frac{\partial x}{\partial s} = 1$, $\frac{\partial y}{\partial t} = 1$. Direct substitution in (2) would yield the symbol $\frac{\partial f}{\partial x}$ in two different senses. On the left of the equation it means the partial derivative of f with respect to x when y is constant, and attention is given to the fact that z is a function of x. On the right of the equation it means the partial derivative of f with respect to x on the assumption that both y and z are constant. Ambiguity is avoided by the use of subscripts as suggested at the close of Ex. 2. Thus we have

$$\left(\frac{\partial f}{\partial x}\right)_y = \left(\frac{\partial f}{\partial x}\right)_{yz} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x}.$$

114. Property of the total differential. An important property of the total differential is expressed in the following theorem:

If $f(x, y, \cdots, z) = c$ for all values of the independent variables, then $df = 0$.

1. Let us suppose first that $x, y, \cdots, z$ are the independent variables. Then $f(x, y, \cdots, z) = c$ for all values of $x, y, \cdots, z$. Hence

$$f(x + \Delta x, y, \cdots, z) - f(x, y, \cdots, z) = 0,$$

and

$$\frac{f(x + \Delta x, y, \cdots, z) - f(x, y, \cdots, z)}{\Delta x} = 0,$$

for all values of x. Taking the limit as $\Delta x \doteq 0$, we have $\frac{\partial f}{\partial x} = 0$. Similarly, $\frac{\partial f}{\partial y} = 0, \cdots, \frac{\partial f}{\partial z} = 0$, and hence

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \cdots + \frac{\partial f}{\partial z}dz = 0.$$

2. Suppose, secondly, that $x, y, \cdots, z$ are functions of the independent variables $s, t, \cdots, u$. Then if $x, y, \cdots, z$ are replaced in $f(x, y, \cdots, z)$ by their values in terms of $s, t, \cdots, u$, we have the first case again, and hence as before $\frac{\partial f}{\partial s} = 0$, $\frac{\partial f}{\partial t} = 0, \cdots, \frac{\partial f}{\partial u} = 0$.

Hence
$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \cdots + \frac{\partial f}{\partial z}dz$$
$$= \frac{\partial f}{\partial s}ds + \frac{\partial f}{\partial t}dt + \cdots + \frac{\partial f}{\partial u}du = 0.$$

It is to be emphasized that the coefficients of dx, dy, $\cdots$, dz are not equal to zero.

115. Implicit functions. CASE I. $f(x, y) = 0$. The equation $f(x, y) = 0$ defines y as an implicit function of x, or x as an implicit function of y, since if one of the variables is given, the values of the other are determined. Then, by § 114,

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0,$$

whence
$$\frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}. \tag{1}$$

Ex. 1. Find the tangent and the normal to the curve $f(x, y) = 0$.

By I, §§ 100, 101, the equations of the tangent and the normal are respectively

$$y - y_1 = \left(\frac{dy}{dx}\right)_1 (x - x_1)$$

and
$$y - y_1 = -\left(\frac{dx}{dy}\right)_1 (x - x_1).$$

By use of (1) these equations become

$$(x - x_1)\left(\frac{\partial f}{\partial x}\right)_1 + (y - y_1)\left(\frac{\partial f}{\partial y}\right)_1 = 0$$

and
$$\frac{(x - x_1)}{\left(\dfrac{\partial f}{\partial x}\right)_1} = \frac{(y - y_1)}{\left(\dfrac{\partial f}{\partial y}\right)_1}.$$

CASE II. $f(x, y, z) = 0$. The equation $f(x, y, z) = 0$ defines any one of the variables x, y, z as a function of the other two. We will take x and y as the independent variables. Then, by § 114,

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0.$$

But
$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

Therefore
$$\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y}\right)dy = 0.$$

This is true for all values of the independent differentials dx and dy. Therefore

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial x}=0 \quad \text{and} \quad \frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial y}=0,$$

whence

$$\frac{\partial z}{\partial x}=-\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial z}}, \qquad \frac{\partial z}{\partial y}=-\frac{\dfrac{\partial f}{\partial y}}{\dfrac{\partial f}{\partial z}}. \tag{2}$$

Ex. 2. Find the equations of the tangent plane and the normal line to the surface $F(x, y, z) = 0$.

By § 112 the required equations are

$$(x-x_1)\left(\frac{\partial z}{\partial x}\right)_1+(y-y_1)\left(\frac{\partial z}{\partial y}\right)_1-(z-z_1)=0$$

and

$$\frac{x-x_1}{\left(\dfrac{\partial z}{\partial x}\right)_1}=\frac{y-y_1}{\left(\dfrac{\partial z}{\partial y}\right)_1}=\frac{z-z_1}{-1}.$$

Using (2) and reducing, we have

$$(x-x_1)\left(\frac{\partial F}{\partial x}\right)_1+(y-y_1)\left(\frac{\partial F}{\partial y}\right)_1+(z-z_1)\left(\frac{\partial F}{\partial z}\right)_1=0$$

and

$$\frac{x-x_1}{\left(\dfrac{\partial F}{\partial x}\right)_1}=\frac{y-y_1}{\left(\dfrac{\partial F}{\partial y}\right)_1}=\frac{z-z_1}{\left(\dfrac{\partial F}{\partial z}\right)_1}.$$

CASE III. $f_1(x, y, z)=0$, $f_2(x, y, z)=0$. The two equations $f_1(x, y, z)=0$, $f_2(x, y, z)=0$ taken simultaneously define any two of the variables as functions of the other one.

By § 114,

$$\frac{\partial f_1}{\partial x}dx+\frac{\partial f_1}{\partial y}dy+\frac{\partial f_1}{\partial z}dz=0,$$

$$\frac{\partial f_2}{\partial x}dx+\frac{\partial f_2}{\partial y}dy+\frac{\partial f_2}{\partial z}dz=0.$$

Therefore

$$dx:dy:dz=\begin{vmatrix}\dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial z}\\ \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial z}\end{vmatrix}:\begin{vmatrix}\dfrac{\partial f_1}{\partial z} & \dfrac{\partial f_1}{\partial x}\\ \dfrac{\partial f_2}{\partial z} & \dfrac{\partial f_2}{\partial x}\end{vmatrix}:\begin{vmatrix}\dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y}\\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y}\end{vmatrix}. \tag{3}$$

Ex. 3. Find the equations of the tangent line to the space curve $f_1(x, y, z)=0$, $f_2(x, y, z) = 0$.

By § 105 the required equations are

$$\frac{x - x_1}{dx} = \frac{y - y_1}{dy} = \frac{z - z_1}{dz},$$

where the values of dx, dy, dz given in (3) may be substituted.

116. Higher partial derivatives. The partial derivatives of $f(x, y)$ are themselves functions of x and y which may have partial derivatives called the *second partial derivatives* of $f(x, y)$. They are $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$, $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$, $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$, $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$. But it may be shown that the order of differentiation with respect to x and y is immaterial (§ 117), so that the second partial derivatives are three in number, expressed by the symbols

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{xy},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}.$$

Similarly, the *third partial derivatives* of $f(x, y)$ are four in number, namely:

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial^3 f}{\partial x^3},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial^2}{\partial x^2}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^3 f}{\partial x^2 \partial y},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial^2}{\partial y^2}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^3 f}{\partial x \partial y^2},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial^3 f}{\partial y^3}.$$

So, in general, $\frac{\partial^{p+q} f}{\partial x^p \partial y^q}$ signifies the result of differentiating $f(x, y)$ p times with respect to x and q times with respect to y, the order of differentiating being immaterial.

The extension to any number of variables is obvious.

117. To prove the relation $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ for any particular values $x=a$, $y=b$, consider the expression

$$I = \frac{f(a+h,\, b+k) - f(a,\, b+k) - f(a+h,\, b) + f(a,\, b)}{hk},$$

where for convenience h and k are taken as positive. We shall prove I equal to $\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{\substack{x=\xi_1\\ y=\eta_1}}$ on the one hand, and to $\left(\frac{\partial^2 f}{\partial y \partial x}\right)_{\substack{x=\xi_2\\ y=\eta_2}}$ on the other hand, where (ξ_1, η_1) and (ξ_2, η_2) are two undetermined points within the rectangle of fig. 70.

In the first place, let $f(x+h,\, y) - f(x,\, y) = F(x,\, y)$. Then

$$I = \frac{1}{h}\,\frac{F(a,\, b+k) - F(a,\, b)}{k};$$

whence, by § 30, (2),

$$I = \frac{1}{h} F_y(a,\, \eta_1)$$

$$= \frac{f_y(a+h,\, \eta_1) - f_y(a,\, \eta_1)}{h},$$

where $F_y(a,\, \eta_1) = \left(\frac{\partial F}{\partial y}\right)_{\substack{x=a\\ y=\eta_1}}$. Applying § 30, (2), a second time, we have

$$I = f_{xy}(\xi_1,\, \eta_1).$$

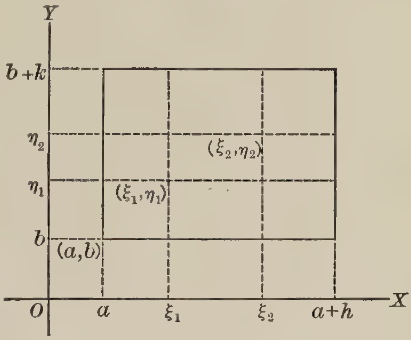

FIG. 70

In the second place, if we let $f(x,\, y+k) - f(x,\, y) = \Phi(x,\, y)$, we have

$$I = \frac{1}{k}\,\frac{\Phi(a+h,\, b) - \Phi(a,\, b)}{h} = \frac{1}{k}\Phi_x(\xi_2,\, b)$$

$$= \frac{f_x(\xi_2,\, b+k) - f_x(\xi_2,\, b)}{k} = f_{yx}(\xi_2,\, \eta_2).$$

Therefore $f_{xy}(\xi_1,\, \eta_1) = f_{yx}(\xi_2,\, \eta_2)$, since each is equal to I.

Now let h and k approach zero as a limit. The points (ξ_1, η_1) and (ξ_2, η_2) both approach the point (a, b) as a limit; and since the functions are continuous

$$\text{Lim}\, f_{xy}(\xi_1,\, \eta_1) = \text{Lim}\, f_{yx}(\xi_2,\, \eta_2),$$

or

$$f_{xy}(a,\, b) = f_{yx}(a,\, b).$$

This result being proved, it is easily extended to any number of variables or differentiations. Thus, for example,

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y \partial x}\right), \text{ whence } \frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial^3 f}{\partial x \partial y \partial x}.$$

118. Higher derivatives of $f(x, y)$ when x and y are functions of t, or of s and t. By § 111, if x and y are functions of t,

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}. \tag{1}$$

To differentiate again with respect to t, we must notice that each term on the right-hand side of (1) is a product and must be handled by the law of products (I, § 96, (4)). We have, then, in the first place

$$\frac{d^2f}{dt^2} = \frac{\partial f}{\partial x}\frac{d^2x}{dt^2} + \frac{dx}{dt}\frac{d}{dt}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial f}{\partial y}\frac{d^2y}{dt^2} + \frac{dy}{dt}\frac{d}{dt}\left(\frac{\partial f}{\partial y}\right).$$

Now $\frac{d}{dt}\left(\frac{\partial f}{\partial x}\right)$ and $\frac{d}{dt}\left(\frac{\partial f}{\partial y}\right)$ may be found from (1) by replacing f by $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ respectively. Hence

$$\begin{aligned}\frac{d^2f}{dt^2} &= \frac{\partial f}{\partial x}\frac{d^2x}{dt^2} + \frac{dx}{dt}\left(\frac{\partial^2 f}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 f}{\partial x\partial y}\frac{dy}{dt}\right) + \frac{\partial f}{\partial y}\frac{d^2y}{dt^2}\\ &\qquad + \frac{dy}{dt}\left(\frac{\partial^2 f}{\partial x\partial y}\frac{dx}{dt} + \frac{\partial^2 f}{\partial y^2}\frac{dy}{dt}\right)\\ &= \frac{\partial^2 f}{\partial x^2}\left(\frac{dx}{dt}\right)^2 + 2\frac{\partial^2 f}{\partial x\partial y}\frac{dx}{dt}\frac{dy}{dt} + \frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dt}\right)^2 + \frac{\partial f}{\partial x}\frac{d^2x}{dt^2} + \frac{\partial f}{\partial y}\frac{d^2y}{dt^2}. \quad (2)\end{aligned}$$

In a similar manner, if x and y are functions of s and t,

$$\begin{aligned}\frac{\partial^2 f}{\partial s^2} &= \frac{\partial^2 f}{\partial x^2}\left(\frac{\partial x}{\partial s}\right)^2 + 2\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial s}\frac{\partial y}{\partial s} + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial y}{\partial s}\right)^2 + \frac{\partial f}{\partial x}\frac{\partial^2 x}{\partial s^2} + \frac{\partial f}{\partial y}\frac{\partial^2 y}{\partial s^2},\\ \frac{\partial^2 f}{\partial t^2} &= \frac{\partial^2 f}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + 2\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\partial f}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial f}{\partial y}\frac{\partial^2 y}{\partial t^2}, \quad (3)\\ \frac{\partial^2 f}{\partial s\partial t} &= \frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial s}\frac{\partial x}{\partial t} + \frac{\partial^2 f}{\partial x\partial y}\left(\frac{\partial x}{\partial s}\frac{\partial y}{\partial t} + \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}\right) + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial s}\frac{\partial y}{\partial t}\\ &\qquad + \frac{\partial f}{\partial x}\frac{\partial^2 x}{\partial s\partial t} + \frac{\partial f}{\partial y}\frac{\partial^2 y}{\partial s\partial t}.\end{aligned}$$

The extension of these formulas to any number of variables is obvious.

Ex. 1. Required to express $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}$, where V is a function of x and y, in polar coördinates.

Since $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$,

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}, \quad \frac{\partial^2 r}{\partial x^2} = \frac{y^2}{(x^2 + y^2)^{\frac{3}{2}}}, \quad \frac{\partial \theta}{\partial x} = \frac{-y}{x^2 + y^2}, \quad \frac{\partial^2 \theta}{\partial x^2} = \frac{2xy}{(x^2 + y^2)^2}.$$

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}, \quad \frac{\partial^2 r}{\partial y^2} = \frac{x^2}{(x^2 + y^2)^{\frac{3}{2}}}, \quad \frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2}, \quad \frac{\partial^2 \theta}{\partial y^2} = \frac{-2xy}{(x^2 + y^2)^2}.$$

Hence, from (3),

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V}{\partial r^2}\frac{x^2}{x^2 + y^2} - 2\frac{\partial^2 V}{\partial r \partial \theta}\frac{xy}{(x^2 + y^2)^{\frac{3}{2}}} + \frac{\partial^2 V}{\partial \theta^2}\frac{y^2}{(x^2 + y^2)^2} + \frac{\partial V}{\partial r}\frac{y^2}{(x^2 + y^2)^{\frac{3}{2}}} + \frac{\partial V}{\partial \theta}\frac{2xy}{(x^2 + y^2)^2},$$

$$\frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial r^2}\frac{y^2}{x^2 + y^2} + 2\frac{\partial^2 V}{\partial r \partial \theta}\frac{xy}{(x^2 + y^2)^{\frac{3}{2}}} + \frac{\partial^2 V}{\partial \theta^2}\frac{x^2}{(x^2 + y^2)^2} + \frac{\partial V}{\partial r}\frac{x^2}{(x^2 + y^2)^{\frac{3}{2}}} - \frac{\partial V}{\partial \theta}\frac{2xy}{(x^2 + y^2)^2}.$$

Hence
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{x^2 + y^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{\sqrt{x^2 + y^2}}\frac{\partial V}{\partial r}$$
$$= \frac{\partial^2 V}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r}\frac{\partial V}{\partial r}.$$

Ex. 2. If $z = f_1(x + at) + f_2(x - at)$, where f_1 and f_2 are any two functions, show that $\frac{\partial^2 z}{\partial t^2} = a^2\frac{\partial^2 z}{\partial x^2}$.

Let $x + at = u$, $x - at = v$; then $\frac{\partial u}{\partial x} = 1$, $\frac{\partial u}{\partial t} = a$, $\frac{\partial v}{\partial x} = 1$, $\frac{\partial v}{\partial t} = -a$, and

$$\frac{\partial z}{\partial x} = \frac{df_1}{du}\frac{\partial u}{\partial x} + \frac{df_2}{dv}\frac{\partial v}{\partial x} = \frac{df_1}{du} + \frac{df_2}{dv},$$

$$\frac{\partial z}{\partial t} = \frac{df_1}{du}\frac{\partial u}{\partial t} + \frac{df_2}{dv}\frac{\partial v}{\partial t} = a\frac{df_1}{du} - a\frac{df_2}{dv}.$$

Differentiating these equations a second time, we have

$$\frac{\partial^2 z}{\partial x^2} = \frac{d^2 f_1}{du^2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{d^2 f_2}{dv^2}\left(\frac{\partial v}{\partial x}\right)^2 = \frac{d^2 f_1}{du^2} + \frac{d^2 f_2}{dv^2},$$

$$\frac{\partial^2 z}{\partial t^2} = \frac{d^2 f_1}{du^2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{d^2 f_2}{dv^2}\left(\frac{\partial v}{\partial t}\right)^2 = a^2\frac{d^2 f_1}{du^2} + a^2\frac{d^2 f_2}{dv^2}.$$

By inspection the required result is obtained.

119. Differentiation of a definite integral. Consider

$$u=\int_a^b f(x, \alpha)\,dx,$$

where α is a parameter independent of x. In the integration α is considered as constant, but the value of the integral is a function of α. Let α be given an increment $\Delta\alpha$. Then u takes an increment Δu, where

$$\begin{aligned}\Delta u &= \int_a^b f(x, \alpha+\Delta\alpha)\,dx-\int_a^b f(x, \alpha)\,dx\\ &= \int_a^b [f(x, \alpha+\Delta\alpha)-f(x, \alpha)]\,dx.\end{aligned}$$

Now by § 31, (1),

$$f(x, \alpha+\Delta\alpha)=f(x, \alpha)+\Delta\alpha\frac{\partial f(x, \alpha)}{\partial\alpha}+\frac{(\Delta\alpha)^2}{2}\frac{\partial^2 f(x, \xi)}{\partial\alpha^2}.$$

Hence
$$\begin{aligned}\Delta u &= \int_a^b\left[\Delta\alpha\frac{\partial f(x, \alpha)}{\partial\alpha}+\frac{(\Delta\alpha)^2}{2}\frac{\partial^2 f(x, \xi)}{\partial\alpha^2}\right]dx\\ &= \Delta\alpha\int_a^b\frac{\partial f(x, \alpha)}{\partial\alpha}\,dx+\frac{(\Delta\alpha)^2}{2}\int_a^b\frac{\partial^2 f(x, \xi)}{\partial\alpha^2}\,dx.\end{aligned}$$

Dividing by $\Delta\alpha$, and taking the limit as $\Delta\alpha$ approaches zero as a limit, we have

$$\frac{\partial u}{\partial\alpha}=\int_a^b\frac{\partial f(x, \alpha)}{\partial\alpha}\,dx. \tag{1}$$

The proof assumes that a and b are finite. It is not always possible to differentiate in this way an integral with an infinite limit. The discussion of this lies outside the scope of this book.

The integral u is also a function of the upper limit b, and we have, by § 25,

$$\frac{\partial u}{\partial b}=f(b, \alpha). \tag{2}$$

Similarly, since $\int_a^b f(x, \alpha)\,dx=-\int_b^a f(x, \alpha)\,dx$,

$$\frac{\partial u}{\partial a}=-f(a, \alpha). \tag{3}$$

Suppose now that a, b, α are all functions of a single variable t. Then, by § 111, (3),

$$\frac{du}{dt}=\frac{\partial u}{\partial b}\frac{db}{dt}+\frac{\partial u}{\partial a}\frac{da}{dt}+\frac{\partial u}{\partial \alpha}\frac{d\alpha}{dt}$$

$$=f(b,\alpha)\frac{db}{dt}-f(a,\alpha)\frac{da}{dt}+\frac{d\alpha}{dt}\int_a^b\frac{\partial f(x,\alpha)}{\partial \alpha}dx. \qquad (4)$$

Ex. If $u=\int_0^\pi \log(1-2a\cos x+a^2)\,dx$,

$$\frac{du}{da}=\int_0^\pi\frac{-2\cos x+2a}{1-2a\cos x+a^2}dx$$

$$=\frac{1}{a}\int_0^\pi\left[1-\frac{1-a^2}{1-2a\cos x+a^2}\right]dx$$

$$=\frac{\pi}{a}-\frac{1-a^2}{a}\int_0^\pi\frac{\sec^2\frac{x}{2}}{(1-a)^2+(1+a)^2\tan^2\frac{x}{2}}dx$$

$$=\frac{\pi}{a}-\frac{2}{a}\left[\tan^{-1}\left(\frac{1+a}{1-a}\tan\frac{x}{2}\right)\right]_0^\pi$$

$$=0.$$

Therefore $u=$ const. But when $a=0$, $u=\int_0^\pi(\log 1)\,dx=0$. Therefore $u=0$.

In this way the value of a definite integral can sometimes be found when direct integration is inconvenient or impossible.

PROBLEMS

1. Given $z=\log(x^2+y^2)$, prove $y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=0$.

2. Given $z=x^4+x^2y^2+y^4$, prove $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=4z$.

3. Given $x^2+y^2-2xy+2z^2=c$, prove $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=0$.

4. Given $y=e^{ax}\sin bx$, prove $x\frac{\partial y}{\partial x}=a\frac{\partial y}{\partial a}+b\frac{\partial y}{\partial b}$.

5. Given $z=(x^2+y^2)\tan^{-1}\frac{y}{x}$, prove $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z$.

6. Given $z^2=xy+\tan^{-1}\frac{y}{x}$, prove $zx\frac{\partial z}{\partial x}+zy\frac{\partial z}{\partial y}=xy$.

7. Given $z=e^y\sin^{-1}(x-y)$, prove $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=z$.

8. Given $z=y^2+2ye^{\frac{1}{x}}$, prove $x^2\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2y^2$.

9. If $z = xy$, illustrate the difference between Δz and dz by constructing a rectangle with sides x and y.

10. A triangle has two of its sides 6 and 8 in. respectively, and the included angle is 30°. Find the change in the area caused by increasing the length of each of the given sides by .01 in. and the included angle by 1°, and compare with the differential of area corresponding to the same increments.

11. A triangle has two of its sides 8 and 12 in. respectively, and the included angle is 60°. Find the change in the opposite side caused by making the given sides 7.9 and 12.1 in., the angle being unchanged, and compare with the differential corresponding to the same increments.

12. A right circular cylinder has an altitude 12 ft. and the radius of its base is 3 ft. Find the change in its volume caused by increasing the altitude by .1 ft. and the radius by .01 ft., and compare with the differential of volume corresponding to the same increments.

13. The distance of an inaccessible object A from a point B is found by measuring a base line $BC = h$ and the angles $CBA = \alpha$ and $BCA = \beta$. Find the expression for the error in the length of AB caused by errors of dh, $d\alpha$, $d\beta$ in measuring h, α, β, assuming that higher powers of the errors of measurement may be neglected.

Verify the formula $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$, in each of the following cases:

14. $z = x^2 + y^2$, $x = t$, $y = t^3$.

15. $z = \sin xy$, $x = e^{2t}$, $y = e^{3t}$.

16. $z = \dfrac{x}{y}$, $x = \sin t$, $y = \cos t$.

17. $z = e^{x^2+y^2}$, $x = \sin t$, $y = \cos t$.

18. Find the tangent plane to the cone $z = a - \sqrt{x^2 + y^2}$.

19. Show that the tetrahedron formed by the coördinate planes and any tangent plane to the surface $xyz = a^3$ is of constant volume.

20. Show that any tangent plane to the surface $z = kxy$ cuts the surface in two straight lines.

21. Find the point in the plane $ax + by + cz + d = 0$ which is nearest the origin.

22. Find the points on the surface $xyz = a^3$ which are nearest the origin.

23. Find a point in a triangle such that the sum of the squares of its distances from the three vertices is a minimum.

24. Of all rectangular parallelopipeds inscribed in an ellipsoid find that which has the greatest volume.

25. Find the point inside a plane triangle from which the sum of the squares of the perpendiculars to the three sides is a minimum. (Express the answer in terms of K, the area of the triangle, a, b, c, the lengths of the three sides, and x, y, z, the three perpendiculars on the sides.)

26. If $z = f\left(\frac{x}{y}\right)$, prove $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$

27. If $f(lx + my + nz,\ x^2 + y^2 + z^2) = 0$, prove

$$(ly - mx) + (ny - mz)\frac{\partial z}{\partial x} + (lz - nx)\frac{\partial z}{\partial y} = 0.$$

28. If $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$, prove $\frac{\partial z}{\partial y} = 2y - \frac{x^2}{y}\frac{\partial z}{\partial x}.$

29. If $f(x, y)$ is a homogeneous function of degree n, prove $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf.$

30. Given $x = r\frac{e^{\theta} + e^{-\theta}}{2}$, $y = r\frac{e^{\theta} - e^{-\theta}}{2}$, prove

$$\left(\frac{\partial x}{\partial r}\right)_\theta = \left(\frac{\partial r}{\partial x}\right)_y, \qquad \left(\frac{\partial y}{\partial r}\right)_\theta = -\left(\frac{\partial r}{\partial y}\right)_x,$$

$$\left(\frac{\partial x}{\partial \theta}\right)_r = -r^2\left(\frac{\partial \theta}{\partial x}\right)_y, \quad \left(\frac{\partial y}{\partial \theta}\right)_r = r^2\left(\frac{\partial \theta}{\partial y}\right)_x.$$

31. Given $u = \log\sqrt{x^2 + y^2}$, $v = \tan^{-1}\frac{y}{x}$, prove

$$\left(\frac{\partial u}{\partial x}\right)_y\left(\frac{\partial x}{\partial u}\right)_v + \left(\frac{\partial u}{\partial y}\right)_x\left(\frac{\partial y}{\partial u}\right)_v = 1,$$

$$\left(\frac{\partial v}{\partial x}\right)_y\left(\frac{\partial x}{\partial v}\right)_u + \left(\frac{\partial v}{\partial y}\right)_x\left(\frac{\partial y}{\partial v}\right)_u = 1.$$

32. Find the tangent plane to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ at the point (x_1, y_1, z_1).

33. Show that the sum of the squares of the intercepts on the coördinate axes of any tangent plane to $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}$ is constant.

34. Show that the sum of the intercepts on the coördinate axes of any tangent plane to $x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}} = a^{\frac{1}{2}}$ is constant.

35. Prove that the plane $lx + my + nz = p$ is tangent to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ if $p = \sqrt{a^2l^2 + b^2m^2 + c^2n^2}$.

36. Prove that the plane $lx + my + nz = p$ is tangent to the paraboloid $ax^2 + by^2 = z$ if $p = -\frac{bl^2 + am^2}{4\,abn}$.

37. Find the cosine of the angle between the normal to the ellipsoid and the straight line drawn from the center to the point of contact, and prove that it is equal to $\frac{p}{r}$, where p is the distance of the tangent plane from the center and r the distance of the point of contact from the center.

38. Find the angle between the line drawn from the origin to the point (a, a, a) of the surface $xyz = a^3$ and the normal to the surface at that point.

39. Find the angle of intersection of the spheres $x^2 + y^2 + z^2 = a^2$ and $(x - b)^2 + y^2 + z^2 = c^2$.

40. Prove that the families of surfaces $x^2y^2 + z^2x^2 = c_1$ and $x^2 - y^2 - z^2 = c_2$ intersect everywhere at right angles.

41. Derive the condition that two surfaces $f(x, y, z) = 0$ and $\phi(x, y, z) = 0$ intersect at right angles.

42. If $f(x, y, z) = 0$, show that $\left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x = -1.$

43. If $f(x, y) = 0$ and $\phi(x, z) = 0$, and z is taken for the independent variable, show that $\frac{\partial \phi}{\partial x}\frac{\partial f}{\partial y}\frac{dy}{dz} = \frac{\partial f}{\partial x}\frac{\partial \phi}{\partial z}.$

44. Find the equations of the tangent line to the curve of intersection of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and the plane $lx + my + nz = 0$.

45. Find the equations of the tangent line to the curve of intersection of the cylinders $x^2 + y^2 = a^2$, $y^2 + z^2 = b^2$.

46. Find the highest point of the curve of intersection of $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and $lx + my + nz = 0$.

47. Find the highest point of the curve of intersection of the hyperboloid $x^2 + y^2 - z^2 = 1$ and the plane $x + y + 2z = 0$.

48. Find the angle at which the helix $x^2 + y^2 = a^2$, $z = k \tan^{-1}\frac{y}{x}$ intersects the sphere $x^2 + y^2 + z^2 = r^2\, (r > a)$.

49. Find the angle at which the curve $y^2 - z^2 = a$, $x = b(y + z)$ intersects the surface $x^2 + 2zy = c$.

Verify $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ in each of the following cases:

50. $z = \frac{x - y}{x + y}$.

51. $z = \log\sqrt{x^2 + y^2}$.

52. $z = \log(x + \sqrt{x^2 + y^2})$.

53. $z = \sin^{-1}\frac{y}{x}$.

54. $z = e^x \sin y$.

55. If $z = \log(x^2 + y^2) + \tan^{-1}\frac{y}{x}$, prove $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$

56. If $z = (e^x - e^{-x})\cos y$, prove $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$

57. If $z = \sec(x - at) + \tan(x + at)$, prove $\frac{\partial^2 z}{\partial t^2} = a^2\frac{\partial^2 z}{\partial x^2}.$

58. If $z = \sqrt{x - y^2}$, prove $\frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial z}{\partial y}\frac{\partial^2 z}{\partial x^2}.$

59. If $z = \sin y + e^{-y}\cos(x - y)$, prove $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} = 0.$

60. If $z = \frac{x^2y^2}{4} + \log x - e^{y^2}$, prove $\frac{\partial^2 z}{\partial x \partial y} = xy.$

61. Given $x = e^u \cos v$, $y = e^u \sin v$, find $\frac{\partial^2 V}{\partial u \partial v}$ in terms of the derivatives of V with respect to x and y.

62. Given $x = e^u \cos v$, $y = e^u \sin v$, prove $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = e^{-2u}\left(\dfrac{\partial^2 V}{\partial u^2} + \dfrac{\partial^2 V}{\partial v^2}\right)$.

63. Given $x = u + v$, $y = \dfrac{u - v}{a}$, prove $a^2 \dfrac{\partial^2 V}{\partial x^2} - \dfrac{\partial^2 V}{\partial y^2} = a^2 \dfrac{\partial^2 V}{\partial u \partial v}$.

64. Given $x = r\dfrac{e^\theta + e^{-\theta}}{2}$, $y = r\dfrac{e^\theta - e^{-\theta}}{2}$, prove

$$\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r}\frac{\partial V}{\partial r}.$$

65. If $x = f(u, v)$ and $y = \phi(u, v)$ are two functions which satisfy the equations $\dfrac{\partial f}{\partial u} = \dfrac{\partial \phi}{\partial v}$, $\dfrac{\partial f}{\partial v} = -\dfrac{\partial \phi}{\partial u}$, and V is any function of x and y, prove

$$\frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial v^2} = \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right)\left[\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right].$$

66. If $z = x\phi\left(\dfrac{y}{x}\right) + \psi\left(\dfrac{y}{x}\right)$, prove $x^2 \dfrac{\partial^2 z}{\partial x^2} + 2xy\dfrac{\partial^2 z}{\partial x \partial y} + y^2 \dfrac{\partial^2 z}{\partial y^2} = 0$.

67. If $z = \phi(x + iy) + \psi(x - iy)$, where $i = \sqrt{-1}$, prove $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$.

68. If $u = f(x, y)$ and $y = F(x)$, find $\dfrac{d^2 u}{dx^2}$.

69. If $f(x, y) = 0$, prove $\dfrac{d^2 y}{dx^2} = -\dfrac{\dfrac{\partial^2 f}{\partial x^2}\left(\dfrac{\partial f}{\partial y}\right)^2 - 2\dfrac{\partial^2 f}{\partial x \partial y}\dfrac{\partial f}{\partial x}\dfrac{\partial f}{\partial y} + \dfrac{\partial^2 f}{\partial y^2}\left(\dfrac{\partial f}{\partial x}\right)^2}{\left(\dfrac{\partial f}{\partial y}\right)^3}$.

70. Differentiate $u = \displaystyle\int_0^\pi \log(1 + a\cos x)\,dx$ with respect to a, and thence find the value of the integral.

71. Differentiate $\displaystyle\int_0^1 \frac{x^a - 1}{\log x}\,dx$ with respect to a, and thence find the value of the integral.

CHAPTER XII

MULTIPLE INTEGRALS

120. Double integral with constant limits. Let $f(x, y)$ be any function of x and y which is single valued and continuous and positive for all pairs of values of x and y for which $a \leqq x \leqq b$ and $c \leqq y \leqq d$. Divide the interval $b-a$ into n equal parts, denoting each part by Δx, thus forming the series of values of $x, x_1, x_2, x_3, \cdots, x_{n-1}$, where

$$x_1 = a + \Delta x, \quad x_2 = x_1 + \Delta x, \quad \cdots, \quad b = x_{n-1} + \Delta x.$$

Similarly, divide the interval $d-c$ into m equal parts, denoting each part by Δy, thus forming the series of values of $y, y_1, y_2, y_3, \cdots, y_{m-1}$, where

$$y_1 = c + \Delta y, \quad y_2 = y_1 + \Delta y, \quad \cdots, \quad d = y_{m-1} + \Delta y.$$

The above values of x determine a series of straight lines parallel to OY, and the values of y determine a series of straight lines parallel to OX (fig. 71).

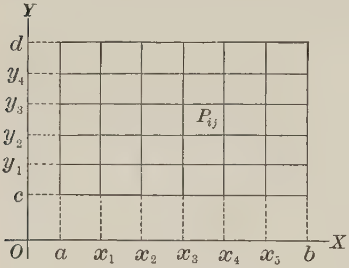

Fig. 71

Every line of either set intersects every line of the other set, and any one of these points of intersection may be represented by $P_{ij}(x_i, y_j)$, where i has all integral values from 0 to $n-1$ and j has all integral values from 0 to $m-1$, x_0 being a and y_0 being c.

Taking the value of $f(x, y)$ at each point of intersection, we form the series

$$\begin{aligned}
&f(a, c)\Delta x\Delta y + f(a, y_1)\Delta x\Delta y + \cdots + f(a, y_{m-1})\Delta x\Delta y \\
&+ f(x_1, c)\Delta x\Delta y + f(x_1, y_1)\Delta x\Delta y + \cdots + f(x_1, y_{m-1})\Delta x\Delta y \\
&+ f(x_2, c)\Delta x\Delta y + f(x_2, y_1)\Delta x\Delta y + \cdots + f(x_2, y_{m-1})\Delta x\Delta y \\
&+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
&+ f(x_{n-1}, c)\Delta x\Delta y + f(x_{n-1}, y_1)\Delta x\Delta y + \cdots + f(x_{n-1}, y_{m-1})\Delta x\Delta y. \quad (1)
\end{aligned}$$

This series can be expressed more concisely by the notation

$$\sum_{i=0}^{i=n-1} \sum_{j=0}^{j=m-1} f(x_i, y_j)\,\Delta x \Delta y, \tag{2}$$

where two $\sum$'s are used, since there are two elements i and j which vary.

The limit of (2) as m and n are both increased indefinitely is called the *double integral* of $f(x, y)$ over the area bounded by the lines $x = a$, $x = b$, $y = c$, and $y = d$.

The summation of the terms in (2) may evidently be made in many ways, but there are two which we shall consider in particular: (1) when the sum of the terms of each row is found, and these sums added together; (2) when the sum of the terms in each column is found, and these sums added together. It will appear from the graphical representation (§ 121) that these two methods lead to the same result; and it may be shown that the result is always independent of the order of summation.

If the first method is followed, it is to be noted that the value of x is the same in all the terms of any one row, and hence each row is exactly the series used in defining a definite integral (§ 21) with y as the independent variable. Accordingly, when we let m increase indefinitely, (2) becomes

$$\begin{aligned}\left(\int_c^d f(a, y)\,dy\right)\Delta x + \left(\int_c^d f(x_1, y)\,dy\right)\Delta x \\ + \left(\int_c^d f(x_2, y)\,dy\right)\Delta x + \cdots + \left(\int_c^d f(x_{n-1}, y)\,dy\right)\Delta x. \end{aligned} \tag{3}$$

But (3) is the series used in defining a definite integral with x as the independent variable and $\int_c^d f(x, y)\,dy$ as the function of x. Letting n increase indefinitely, we have

$$\int_a^b \left(\int_c^d f(x, y)\,dy\right) dx, \tag{4}$$

which represents the double integral on the hypothesis that first m and then n is made to increase indefinitely.

Another way of writing (4) is

$$\int_a^b \int_c^d f(x, y)\, dx dy, \tag{5}$$

where the summation is made in the order of the differentials from right to left, i.e. first with respect to y and then with respect to x, and the limits are in the same order as the differentials, i.e. the limits of y are c and d, and the limits of x are a and b.*

If the second method of summation is followed, we have

$$\int_c^d \left(\int_a^b f(x, y)\, dx\right) dy, \tag{6}$$

or

$$\int_c^d \int_a^b f(x, y)\, dy dx. \tag{7}$$

Ex. The moment of inertia of a particle about an axis is the product of its mass by the square of its distance from the axis. From this definition let us determine the moment of inertia of a lamina of uniform thickness k about an axis perpendicular to its plane. Let the density of the lamina be uniform and denoted by ρ, and let the plane XOY coincide with the plane of the lamina, the axis being perpendicular to the plane at O. Let the lamina be in the form of a rectangle (fig. 72) bounded by the lines $x = a$, $x = b$, $y = c$, $y = d$. Divide the lamina into rectangles by the lines

$$x = x_1, \quad x = x_2, \quad x = x_3, \quad \cdots,$$
$$y = y_1, \quad y = y_2, \quad y = y_3, \quad \cdots.$$

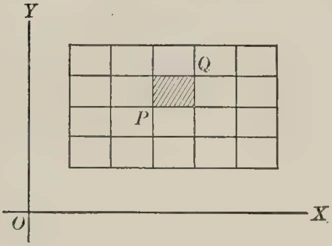

Fig. 72

Then the mass of any element, as PQ, is $\rho k \Delta x \Delta y$. If this mass is regarded as concentrated at $P(x_i, y_j)$, its moment of inertia would be $(x_i^2 + y_j^2)\rho k \Delta x \Delta y$. If the mass is regarded as concentrated at $Q(x_i + \Delta x, y_j + \Delta y)$, its moment of inertia would be $[(x_i+\Delta x)^2+(y_j+\Delta y)^2]\,\rho k\,\Delta x\Delta y$. Letting M represent the moment of inertia of the rectangle, we have

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* Still another form of writing (4) is $\int_a^b dx \int_c^d f(x, y)\, dy$, in which the order of summation is first with respect to y and then with respect to x.

Some writers also prefer to write (5) in the form $\int_a^b \int_c^d f(x, y)\, dy dx$, which is merely (4) with the parenthesis removed. In this form it is to be noted that the limits and the differentials are in inverse orders, and that the order of summation is the order of the differentials from left to right, i.e. first with respect to y and then with respect to x. In this text this last form of writing the double integral will not be used. In other books the context will indicate the form of notation which the writer has chosen.

$$\sum_{i=0}^{i=n-1}\sum_{j=0}^{j=m-1}(x_i^2+y_j^2)\rho k\Delta x\Delta y<M<\sum_{i=0}^{i=n-1}\sum_{j=0}^{j=m-1}[(x_i+\Delta x)^2+(y_j+\Delta y)^2]\rho k\Delta x\Delta y.$$

The limits of these sums as $n=\infty$, $m=\infty$, and $\Delta x\doteq 0$, $\Delta y\doteq 0$ are the same (§ 3), for

$$\text{Lim}\,\frac{[(x_i+\Delta x)^2+(y_j+\Delta y)^2]\,\rho k\,\Delta x\Delta y}{(x_i^2+y_j^2)\,\rho k\,\Delta x\Delta y}=1.$$

Hence we define M by the equation

$$M=\int_a^b\int_c^d(x^2+y^2)\,\rho k dx dy.$$

If ρ and k are each placed equal to unity, the result is often referred to as the moment of inertia of the plane area bounded by the lines $x=a$, $x=b$, $y=c$, $y=d$.

121. Graphical representation. Placing $z=f(x, y)$, we have the equation of a surface which is the graphical representation of $f(x, y)$ (fig. 73). Through the lines $x=a$, $x=b$, $y=c$, $y=d$ pass planes parallel to OZ. Then the volume bounded by the

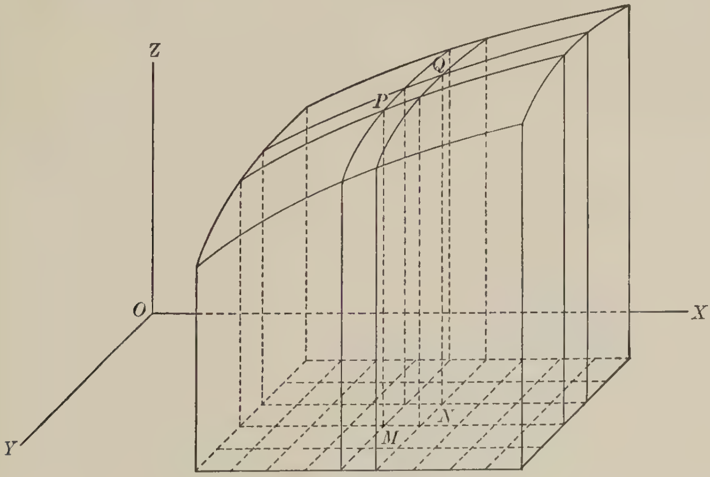

FIG. 73

plane XOY, the planes $x=a$, $x=b$, $y=c$, $y=d$, and the surface $z=f(x, y)$ is a graphical representation of the double integral of § 120. For if the planes $x=x_1$, $x=x_2$, $x=x_3$, $\cdots$, $y=y_1$, $y=y_2$, $y=y_3$, $\cdots$ are constructed, they divide the above volume, which we will denote by V, into columns such as $MNQP$, each of which stands on a rectangular base $\Delta x \Delta y$ in the plane XOY. If the coördinates of M are (x_i, y_j), the corresponding term of (2), § 120, is $f(x_i, y_j)\Delta x \Delta y$, and since $f(x_i, y_j) = MP$, the term $f(x_i, y_j)\Delta x \Delta y$ is the volume of a prism standing on the same base as the column $MNQP$, and the volume of this prism is approximately the volume of the column. Hence (2), § 120, is the sum of the volumes of such prisms, and is approximately equal to V; and as m and n both increase indefinitely, the limit of the sum of the volumes of these prisms is evidently V.

The significance of the two ways of summation is now clear. For if the integral is written as (5), § 120, the prisms are first added together, keeping x constant, the result being a series of slices, each of thickness Δx, which are finally added together to include the total volume; and if the integral is written in the form (7), § 120, the prisms are first added together, keeping y constant, the result being a series of slices, each of thickness Δy, which are finally added together to include the total volume. It follows that (5) and (7) are equivalent, as was noted before.

122. Double integral with variable limits. We may now extend the idea of a double integral as follows: Instead of taking the integral over a rectangle, as in § 120, we may take it over an area bounded by any closed curve (fig. 74) such that a straight line parallel to either OX or OY intersects it in not more than two points. Drawing straight lines parallel to OY and straight lines parallel to OX, we form rectangles of area $\Delta x \Delta y$, some of which are entirely within the area bounded by the curve and others of which are only partly within that area. Then

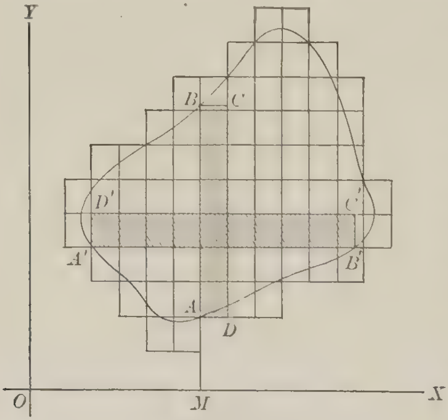

Fig. 74

$$\sum\sum f(x, y)\Delta x \Delta y, \qquad (1)$$

where the summation includes all the rectangles which are wholly or partly within the curve, represents approximately the volume bounded by the plane XOY, the surface $z=f(x, y)$, and the cylinder standing on the curve as a base, since it is the sum of the volumes of prisms, as in § 121. Now, letting the number of these prisms increase indefinitely, while $\Delta x \doteq 0$ and $\Delta y \doteq 0$, it is evident that (1) approaches a definite limit, the volume described above.

If we sum up first with respect to y, we add together terms of (1) corresponding to a fixed value of x, such as x_i. Then if MB is the line $x=x_i$, the result is a sum corresponding to the strip $ABCD$, and the limits of y for this strip are the values of y corresponding to $x=x_i$ in the equation of the curve; i.e. if $MA=f_1(x_i)$ and $MB=f_2(x_i)$, the limits of y are $f_1(x_i)$ and $f_2(x_i)$. As different integral values are given to i, we have a series of terms corresponding to strips of the type $ABCD$, which, when the final summation is made with respect to x, must cover the area bounded by the curve. Hence, if the least and the greatest values of x for the curve are the constants a and b respectively, the limit of (1) appears in the form

$$\int_a^b \int_{f_1(x)}^{f_2(x)} f(x, y)\, dx dy, \tag{2}$$

where the subscript i is no longer needed.

On the other hand, if the first summation is made with respect to x, the result is a series of terms each of which corresponds to a strip of the type $A'B'C'D'$, and the limits of x are of the form $\phi_1(y)$ and $\phi_2(y)$, found by solving the equation of the curve for x in terms of y. Finally, if the least and the greatest values of y for the curve are the constants c and d respectively, the limit of (1) appears in the form

$$\int_c^d \int_{\phi_1(y)}^{\phi_2(y)} f(x, y)\, dy dx. \tag{3}$$

While the limits of integration in (2) and (3) are different, it is evident from the graphical representation that the integrals are equivalent.

123. In §§ 120–122, $f(x, y)$ has been assumed positive for all the values of x and y considered, i.e. the surface $z=f(x, y)$ was entirely on the positive side of the plane XOY. If, however,

$f(x, y)$ is negative for all the values of x and y considered, the reasoning is exactly as in the first case, but the value of the integral is negative. Finally, if $f(x, y)$ is sometimes positive and sometimes negative, the result is an algebraic sum, as in § 22. Furthermore, it is not necessary that all the values of Δx should be equal and all the values of Δy equal; also in place of $f(x_i, y_j)$ we may use $f(\xi_i, \eta_j)$, where $x_i < \xi_i < x_{i+1}$ and $y_j < \eta_j < y_{j+1}$.

The work of making these extensions being similar to that of §§ 22–23, it is not repeated here, but the student is advised to review those articles.

124. Computation of a double integral. The method of computing a double integral is evident from the meaning of the notation.

Ex. 1. Find the value of $\int_0^3 \int_0^2 xydxdy$.

As this integral is written, it is equivalent to $\int_0^3 \left(\int_0^2 xydy \right) dx$, the integral in parenthesis being computed first, on the hypothesis that y alone varies.

$$\int_0^2 xydy = \left[\frac{xy^2}{2} \right]_0^2 = 2x.$$

$$\int_0^3 2xdx = [x^2]_0^3 = 9.$$

Ex. 2. Find the value of the integral $\int\int xydxdy$ over the first quadrant of the circle $x^2 + y^2 = a^2$.

If we sum up first with respect to y, we find a series of terms corresponding to strips of the type $ABCD$ (fig. 75), and the limits of y are the ordinates of the points like A and B. The ordinate of A is evidently 0, and from the equation of the circle the ordinate of B is $\sqrt{a^2 - x^2}$, where $OA = x$. Finally, to cover the quadrant of the circle the limits of x are 0 and a. Hence the required integral is

$$\int_0^a \int_0^{\sqrt{a^2 - x^2}} xydxdy = \int_0^a \left[\frac{xy^2}{2} \right]_0^{\sqrt{a^2 - x^2}} dx$$

$$= \int_0^a \frac{x}{2}(a^2 - x^2)\, dx$$

$$= \frac{1}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4} \right]_0^a$$

$$= \tfrac{1}{8} a^4.$$

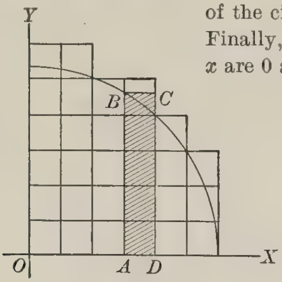

Fig. 75

Since the above computation of a double integral is simply the repeated computation of a single definite integral, the theorems of § 24 may be used in simplifying the work.

125. Double integral in polar coördinates. If we have to find the double integral of $f(r, \theta)$ over any area, we divide that area up into elements, such as $ABCD$ (fig. 76), by drawing radii vectors at distances $\Delta\theta$ apart, and concentric circles the radii of which increase by Δr. The area of $ABCD$ is the difference of the areas of the sectors OBC and OAD. Hence, if $OA = r$,

$$\text{area } ABCD = \tfrac{1}{2}(r+\Delta r)^2\Delta\theta - \tfrac{1}{2}r^2\Delta\theta$$
$$= r\Delta r\Delta\theta + \tfrac{1}{2}\overline{\Delta r}^2\cdot\Delta\theta.$$

Therefore any term of the sum corresponding to (1), § 120, is, at first sight, of the form

$$f(r, \theta)\cdot(r\Delta r\Delta\theta + \tfrac{1}{2}\overline{\Delta r}^2\cdot\Delta\theta).$$

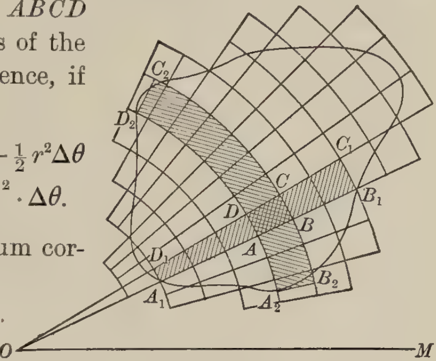

FIG. 76

But in taking the limit, $r\Delta r\Delta\theta + \frac{1}{2}\overline{\Delta r}^2\cdot\Delta\theta$ may be replaced by $r\Delta r\Delta\theta$ (§ 3), for

$$\text{Lim}\,\frac{f(r, \theta)\cdot(r\Delta r\Delta\theta + \frac{1}{2}\overline{\Delta r}^2\cdot\Delta\theta)}{f(r, \theta)\cdot(r\Delta r\Delta\theta)} = \text{Lim}\left(1 + \frac{1}{2}\cdot\frac{\Delta r}{r}\right) = 1.$$

Hence the required integral is

$$\underset{\substack{\Delta r \doteq 0 \\ \Delta\theta \doteq 0}}{\text{Lim}} \sum\sum f(r, \theta)\,r\Delta r\Delta\theta = \iint f(r, \theta)\,r\,dr\,d\theta. \qquad (1)$$

If the summation in (1) is made first with respect to r, the result is a series of terms corresponding to strips such as $A_1B_1C_1D_1$, and the limits of r are functions of θ found from the equation of the bounding curve. The summation with respect to θ will then add all these terms, and the limits of θ taken so as to cover the entire area will be constants, i.e. the least and the greatest values of θ on the bounding curve.

If, on the other hand, the summation is made first with respect to θ, the result is a series of terms corresponding to strips such as $A_2B_2C_2D_2$, and the limits of θ are functions of r found from the equation of the bounding curve. The summation with respect to r will then add all these terms, and the limits of r will be the least and the greatest values of r on the bounding curve.

Ex. Find the integral of r^2 over the circle $r = 2a\cos\theta$.

If we sum up first with respect to r, the limits are 0 and $2a\cos\theta$, found from the equation of the bounding curve, and the result is a series of terms corresponding to sectors of the type AOB (fig. 77). To sum up these terms so as to cover the circle, the limits of θ are $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. The result is

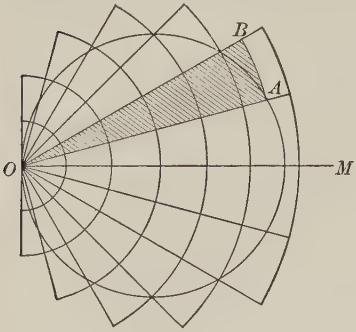

FIG. 77

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2a\cos\theta} r^3 d\theta dr = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\frac{r^4}{4}\right]_0^{2a\cos\theta} d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4a^4\cos^4\theta d\theta$$

$$= \tfrac{3}{2}\pi a^4.$$

The graphical representation may be made by the use of cylindrical coördinates defined in § 127.

126. Triple integrals. Let any volume (fig. 78) be divided into rectangular parallelopipeds of volume $\Delta x\Delta y\Delta z$ by planes parallel respectively to the coördinate planes, some of the parallelopipeds extending outside the volume in a manner similar to that in which the rectangles in § 122 extend outside the area. Let (x_i, y_j, z_k) be a point of intersection of any three of these planes and form the sum

$$\sum_{i=0}^{i=n}\sum_{j=0}^{j=m}\sum_{k=0}^{k=p} f(x_i, y_j, z_k)\Delta x\Delta y\Delta z,$$

as in § 122. Then the limit of this sum as n, m, and p increase indefinitely, while $\Delta x \doteq 0$, $\Delta y \doteq 0$, $\Delta z \doteq 0$, so as to include all points of the volume, is called the triple integral of $f(x, y, z)$ throughout the volume. It is denoted by the symbol

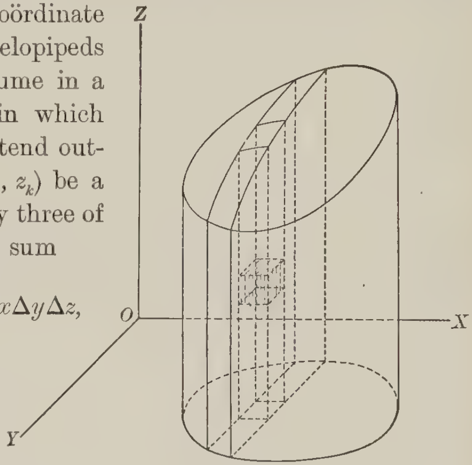

FIG. 78

$$\iiint f(x, y, z)\,dx\,dy\,dz,$$

the limits remaining to be substituted. If the summation is made first with respect to z, x and y remaining constant, the result is to extend the integration throughout a column of cross section $\Delta x \Delta y$; if next x remains constant and y varies, the integration is extended so as to combine the columns into slices; and finally, as x varies, the slices are combined so as to complete the integration throughout the volume.

127. Cylindrical and polar coördinates. In addition to the rectangular coördinates defined in § 84, we shall consider two other systems of coördinates for space of three dimensions, — (1) *cylindrical coördinates*, (2) *polar coördinates*.

1. *Cylindrical coördinates.* If the x and the y of the rectangular coördinates are replaced by polar coördinates r and θ in the plane XOY, and the z coördinate is retained with its original significance, the new coördinates r, θ, and z are called *cylindrical coördinates.* The formulas connecting the two systems of coördinates are evidently

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

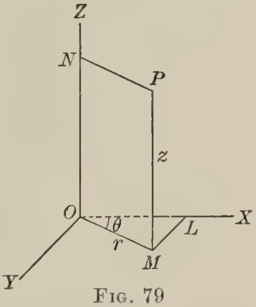

FIG. 79

Turning to fig. 79, we see that $z = z_1$ determines a plane parallel to the plane XOY, that $\theta = \theta_1$ determines a plane $MONP$, passing through OZ and making an angle θ_1 with the plane XOZ, and that $r = r_1$ determines a right circular cylinder with radius r_1 and OZ as its axis. These three surfaces intersect at the point P.

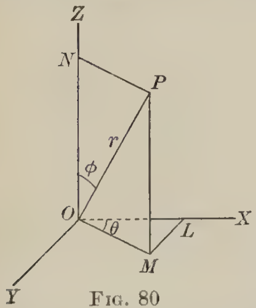

FIG. 80

2. *Polar coördinates.* In fig. 80 the cylindrical coördinates of P are $OM = r$, $MP = z$, and $\angle LOM = \theta$. If instead of placing $OM = r$ we place $OP = r$, and denote the angle NOP by ϕ, we shall have r, ϕ, and θ as the *polar coördinates* of P. Then, since $ON = OP\cos\phi$ and $OM = OP\sin\phi$, the following equations evidently express the connection between the rectangular and the polar coördinates of P:

$$z = r\cos\phi, \quad x = r\sin\phi\cos\theta, \quad y = r\sin\phi\sin\theta.$$

The polar coördinates of a point also determine three surfaces which intersect at the point. For $\theta = \theta_1$ determines a plane (fig. 81) through OZ, making the angle θ_1 with the plane XOZ; $\phi = \phi_1$ determines a cone of revolution, the axis and the vertical angle of which are respectively OZ and $2\phi_1$; and $r = r_1$ determines a sphere with its center at O and radius r_1.

In both systems of coördinates θ varies from 0 to 2π, and in polar coördinates ϕ varies from 0 to π. The coördinate r is usually positive in both systems, but may be negative, in which case it will be laid off on the backward extension of the line determined by the other two coördinates, as in I, § 177.

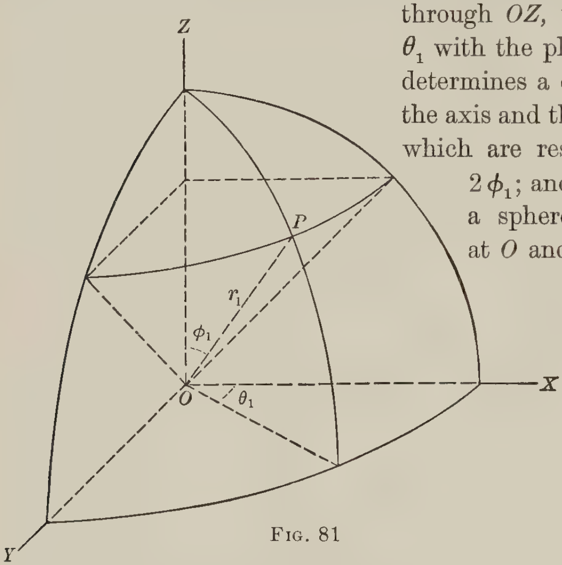

Fig. 81

128. Elements of volume in cylindrical and in polar coördinates. If it is desired to express the triple integral of § 126 in either cylindrical or polar coördinates, it is necessary to know the expression for the element of volume in those coördinates.

1. The element of volume in cylindrical coördinates (fig. 82) is the volume bounded by two cylinders of radii r and $r + \Delta r$, two planes corresponding to z and $z + \Delta z$, and two planes corresponding to θ and $\theta + \Delta\theta$. It is accordingly, except for

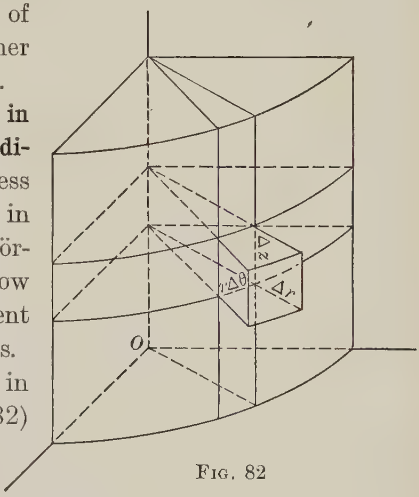

Fig. 82

infinitesimals of higher order, a cylinder with altitude Δz and base $r\Delta r\Delta\theta$ (§ 125). Hence the element of volume is

$$dV = r dr d\theta dz. \tag{1}$$

2. The element of volume in polar coördinates (fig. 83) is the volume bounded by two spheres of radii r and $r+\Delta r$, two conical surfaces corresponding to ϕ and $\phi+\Delta\phi$, and two planes corresponding to θ and $\theta+\Delta\theta$. The volume of the spherical pyramid O–$ABCD$ is equal to the area of its base $ABCD$ multiplied by one third of its altitude r.* To find the area of $ABCD$ we note first that the area of the zone formed by completing the arcs AD and BC is equal to its altitude, $r\cos\phi - r\cos(\phi+\Delta\phi)$, multiplied by $2\pi r$. Also the area of $ABCD$ is to the area of the zone as the angle $\Delta\theta$ is to 2π.

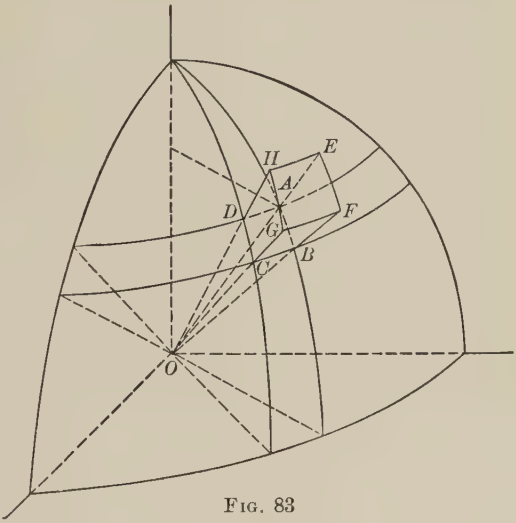

FIG. 83

Hence area $ABCD = r\Delta\theta[r\cos\phi - r\cos(\phi+\Delta\phi)]$,

and vol O–$ABCD = \frac{1}{3}r^3\Delta\theta[\cos\phi - \cos(\phi+\Delta\phi)]$.

Similarly,

$$\text{vol } O\text{–}EFGH = \tfrac{1}{3}(r+\Delta r)^3\Delta\theta[\cos\phi - \cos(\phi+\Delta\phi)].$$

Therefore

$$\text{vol } ABCDEFGH = \tfrac{1}{3}[(r+\Delta r)^3 - r^3]\Delta\theta[\cos\phi - \cos(\phi+\Delta\phi)].$$

But this expression differs from $r^2\sin\phi\Delta r\Delta\phi\Delta\theta$ by an infinitesimal of higher order. Accordingly, the element of volume is

$$dV = r^2\sin\phi dr d\phi d\theta. \tag{2}$$

---

* The volume of a spherical pyramid is to the volume of the sphere as the area of its base is to the area of the surface of the sphere.

It is to be noted that dV is equal to the product of the three dimensions AB, AD, and AE, which are respectively $rd\phi$, $r\sin\phi d\theta$, and dr.

129. Change of coördinates. When a double integral is given in the form $\iint f(x, y)\,dxdy$, where the limits are to be substituted so as to cover a given area, it may be easier to determine the value of the integral if the rectangular coördinates are replaced by polar coördinates. Then $f(x, y)$ becomes $f(r\cos\theta, r\sin\theta)$, i.e. a function of r and θ. As the other factor, $dxdy$, indicates the element of area, in view of the graphical representation of § 121 and the work of § 125, we may replace $dxdy$ by $rdrd\theta$. These two elements of area are not equivalent, but the two integrals are nevertheless equivalent, provided the limits of integration in each system of coördinates are taken so as to cover the same area.

In like manner, the three triple integrals

$$\iiint f(x, y, z)\,dxdydz,$$

$$\iiint f(r\cos\theta, r\sin\theta, z)\,rdrd\theta dz,$$

$$\iiint f(r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)\,r^2\sin\phi drd\phi d\theta$$

are equivalent when the limits are so taken in each as to cover the total volume to be considered.

PROBLEMS

Find the values of the following integrals:

1. $\int_1^2\int_1^{x^2}\frac{x}{y}\,dxdy.$

2. $\int_0^2\int_{\frac{1}{x}}^{x}\frac{x^2}{y^2}\,dxdy.$

3. $\int_{\frac{\pi}{2}}^{\pi}\int_0^{y^2}\sin\frac{x}{y}\,dydx.$

4. $\int_2^4\int_1^{\sqrt{y}} x\log\frac{x^2}{y}\,dydx.$

5. $\int_0^{\pi}\int_0^{a\sqrt{\sin\theta}} r^3 d\theta dr.$

6. $\int_0^{\pi}\int_1^{\cos\theta+2} r^2\sin\theta d\theta dr.$

7. $\int_0^{a}\int_{-\cos^{-1}\frac{r}{a}}^{\cos^{-1}\frac{r}{a}} rdrd\theta.$

8. $\int_0^1\int_0^{\log\frac{1}{x}}\int_0^{y-2x} e^{x-2y+z}\,dxdydz.$

9. $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dxdydz}{\sqrt{a^2-x^2-y^2-z^2}}$.

10. $\int_1^2 \int_0^x \int_0^y \frac{dxdydz}{x^2+y^2}$.

11. $\int_0^{\frac{\pi}{2}} \int_0^{a\cos\theta} \int_0^{r\sin\theta} r\,d\theta drdz$.

12. $\int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{r^2\sin\theta\cos\theta} r^3 drd\theta dz$.

13. $\int_0^{\pi} \int_0^{2\pi} \int_0^a r^4 \sin^3\phi d\phi d\theta dr$.

14. $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_{a\sin\phi}^{a} r\sin\phi\cos\phi\cos\theta d\theta d\phi dr$.

15. Prove that $\int_a^b \int_a^b f(x)\cdot f(y)\,dx\,dy = \left[\int_a^b f(x)\,dx\right]^2$.

CHAPTER XIII

APPLICATIONS OF MULTIPLE INTEGRALS

130. Moment of inertia of a plane area. *The moment of inertia of a particle about an axis is the product of its mass and the square of its distance from the axis.* The moment of inertia of a number of particles about the same axis is the sum of the moments of inertia of the particles about that axis. From this definition we derive (§ 120) a definition of the moment of inertia of a homogeneous rectangular lamina of thickness k and density ρ about an axis perpendicular to the plane of the lamina. The result may be written in the form

$$M = \rho k \int_a^b \int_c^d (x^2 + y^2)\, dx dy, \tag{1}$$

where M represents the moment of inertia.

If ρ and k are both replaced by unity, (1) becomes

$$M = \int_a^b \int_c^d (x^2 + y^2)\, dx dy, \tag{2}$$

which, as was noted in § 120, is called the moment of inertia of the rectangle about an axis perpendicular to its plane at O.

Reasoning in the same way, we may form the general expression

$$\iint (x^2 + y^2)\, dx dy, \tag{3}$$

where the integration is to extend over a given area in the plane XOY. Then (3) is the moment of inertia of that area about the axis perpendicular to the plane at O.

Ex. Find the moment of inertia, about an axis perpendicular to the plane at the origin, of the plane area (fig. 84) bounded by the parabola $y^2 = 4ax$, the line $y = 2a$, and the axis OY.

If the integration is made first with respect to x, the limits of that integration are 0 and $\frac{y^2}{4a}$, since the operation is the summing of elements of moment of inertia due to the elementary rectangles in any strip corresponding to a fixed value of y; the limit 0 is found from the axis of y, and the limit $\frac{y^2}{4a}$ is found from the equation of the parabola.

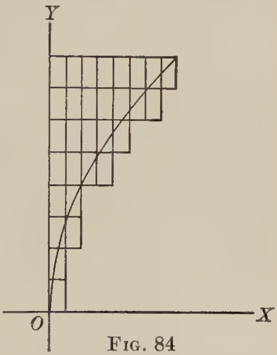

FIG. 84

Finally, the limits of y must be taken so as to include all the strips parallel to OX, and hence must be 0 and $2a$.

$$\text{Therefore } M = \int_0^{2a}\int_0^{\frac{y^2}{4a}}(x^2+y^2)\,dydx$$

$$= \int_0^{2a}\left(\frac{1}{192}\cdot\frac{y^6}{a^3}+\frac{1}{4}\cdot\frac{y^4}{a}\right)dy$$

$$= \tfrac{178}{105}a^4.$$

131. If the plane area is more conveniently defined by means of polar coördinates, (3) of § 130 becomes

$$M = \iint r^2(rdrd\theta); \tag{1}$$

for, by § 129, in place of $dxdy$ as the element of area we take the element of area $rdrd\theta$, and the factor x^2+y^2 evidently becomes r^2.

Formula (1) may also be derived directly from the fundamental definition at the beginning of § 130, and the student is advised to make that derivation.

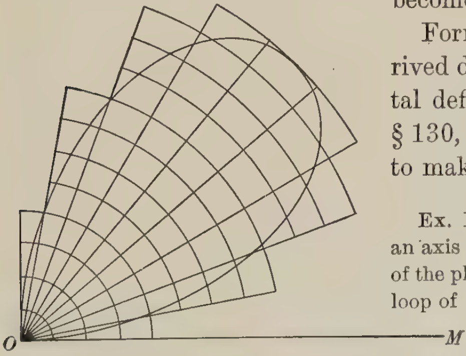

FIG. 85

Ex. Find the moment of inertia, about an axis perpendicular to the plane at O, of the plane area (fig. 85) bounded by one loop of the curve $r = a \sin 2\theta$.

We shall take the loop in the first quadrant, since the moments of inertia of all the loops about the chosen axis are the same by the symmetry of the curve.

If the first integration is made with respect to r, the result is the moment of inertia of a strip bounded by two successive radii vectors and a circular arc; and hence the limits for r are 0 and $a\sin 2\theta$. Since the values of θ for the loop of the curve vary from 0 to $\frac{\pi}{2}$, it is evident that those values are the limits for θ in the final integration.

Therefore
$$M = \int_0^{\frac{\pi}{2}}\int_0^{a\sin 2\theta} r^3\,d\theta dr$$
$$= \tfrac{1}{4}a^4\int_0^{\frac{\pi}{2}}\sin^4 2\theta d\theta$$
$$= \tfrac{3}{64}\pi a^4.$$

132. In the two preceding articles we have found the moment of inertia of a plane area about an axis perpendicular to the plane, which, with the exception of a constant factor, is the moment of inertia of a corresponding homogeneous lamina about the same axis. We shall now find the moment of inertia of a homogeneous lamina about an axis in its plane.

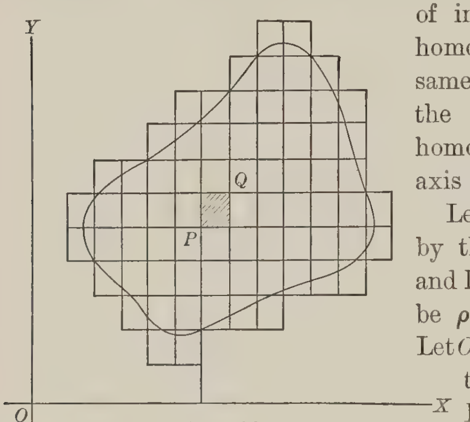

FIG. 86

Let the lamina be bounded by the closed curve (fig. 86), and let its density at any point be ρ and its thickness be k. Let OX be the axis about which the moment is to be taken. Divide the area into rectangles of area $\Delta x\Delta y$. Then the mass of any corresponding element of the lamina, as PQ, is $\rho k\Delta x\Delta y$. If this mass is regarded as concentrated at P, its moment about OX is $\rho k y^2\Delta x\Delta y$; and if the mass is regarded as concentrated at Q, its moment about OX is $\rho k(y+\Delta y)^2\Delta x\Delta y$.

Therefore, if M_x represents the required moment,

$$\sum\sum \rho k y^2\Delta x\Delta y < M_x < \sum\sum \rho k(y+\Delta y)^2\Delta x\Delta y, \qquad (1)$$

the summations to cover the entire area.

Since $\text{Lim}\,\frac{\rho k(y+\Delta y)^2\Delta x\Delta y}{\rho k y^2\Delta x\Delta y}=1$, the double sums of (1) have the same limit (§ 3), and accordingly

$$M_x = \iint \rho k y^2 dx dy, \qquad (2)$$

where the integration is to cover the entire area.

If ρ and k are each replaced by unity, (2) defines the moment of inertia of the plane area about OX.

If M_y denotes the moment of inertia about OY, in similar manner

$$M_y = \iint \rho k x^2 dx dy. \tag{3}$$

Ex. Find the moment of inertia about OY of the plane area bounded by the parabola $y^2 = 4ax$, the line $y = 2a$, and the axis OY.

Since the above area is the same as that of the Ex. in § 130, the limits of integration will be the same as there determined, but the integrand will be changed in that $x^2 + y^2$ is replaced by x^2.

Hence
$$\begin{aligned} M_y &= \int_0^{2a}\int_0^{\frac{y^2}{4a}} x^2 dy dx \\ &= \frac{1}{192\,a^3}\int_0^{2a} y^6 dy \\ &= \tfrac{2}{21}\, a^4. \end{aligned}$$

If it is desirable to use polar coördinates, (2) becomes

$$M_x = \iint \rho k r^3 \sin^2\theta dr d\theta, \tag{4}$$

and (3) becomes

$$M_y = \iint \rho k r^3 \cos^2\theta dr d\theta, \tag{5}$$

the substitution being made according to § 129.

133. Area bounded by a plane curve. The area bounded by an arc of a plane curve, the axis of x, and the ordinates of the ends of the arc has been determined in § 35 by a single integration. By taking the algebraic sum of such areas any plane area may be computed.

The area bounded by any plane curve may also be found by a double integration as follows: Draw straight lines parallel to OX and to OY respectively, forming rectangles of area $\Delta x \Delta y$, some of which, as in fig. 74, will be entirely within the curve, and others of which will be only partly within the curve. Form the double sum $\sum\sum \Delta x \Delta y$ of these rectangles, and then let their number increase indefinitely while $\Delta x \doteq 0$ and $\Delta y \doteq 0$. Then the double integral

$$\iint dx dy \tag{1}$$

is the required area.

Ex. Find the area inclosed by the curve $(y - x - 3)^2 = 4 - x^2$ (fig. 87).

The element of area is the rectangle $\Delta x \Delta y$. If the first integration is made with respect to y, the result is the area of a strip like the one shaded in fig. 87, and the limits for y will be found by solving the equation of the curve for y in terms of x. Since $y = x + 3 \pm \sqrt{4 - x^2}$, the lower limit is $y_1 = x + 3 - \sqrt{4 - x^2}$ and the upper limit is $y_2 = x + 3 + \sqrt{4 - x^2}$. For the integration with respect to x the limits are -2 and 2, since the curve is bounded by the lines $x = -2$ and $x = 2$.

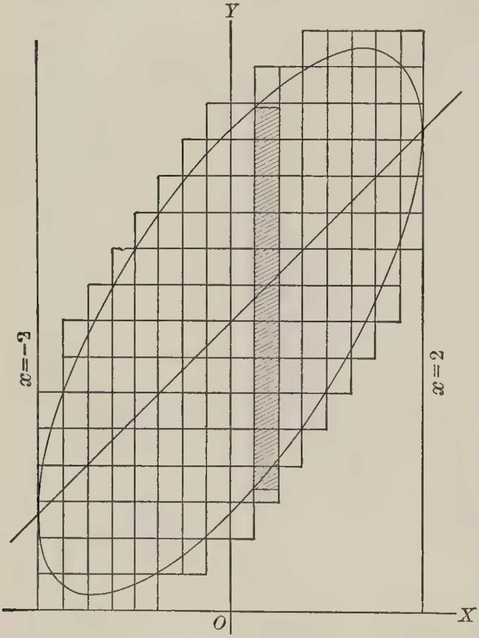

FIG. 87

Therefore

$$\text{area} = \int_{-2}^{2}\int_{y_1}^{y_2} dx dy$$

$$= \int_{-2}^{2} (y_2 - y_1)\, dx$$

$$= 2\int_{-2}^{2} \sqrt{4 - x^2}\, dx$$

$$= 4\pi.$$

This example is Ex. 3, § 35. Comparing the two solutions, we see that the result of the first integration here is exactly the integrand in § 35. It is evident that this will be the case in all similar problems, and hence many areas may be found by single integration. The advantage of the double integral consists in the representation of the area of a figure for which the limits of integration cannot easily be found.

134. In like manner, the area bounded by any curve in polar coördinates may be expressed by the double integral

$$\iint r\, dr d\theta, \tag{1}$$

the element of area being that bounded by two radii vectors the angles of which differ by $\Delta\theta$, and by the arcs of two circles the radii of which differ by Δr.

If the first integration of (1) is with respect to r, the result before the substitution of the limits is $\frac{1}{2} r^2 d\theta$. But this is exactly the integrand used in computation by a single integration. Hence many areas may be computed by single integration in polar coördinates.

135. Area of any surface. Let C (fig. 88) be any closed curve on the surface $f(x, y, z) = 0$. Let its projection on the plane XOY be C'. We shall assume that the given surface is such that the perpendicular to the plane XOY at any point within the curve C' meets the surface in but a single point.

In the plane XOY draw straight lines parallel to OX and OY, forming rectangles of area $\Delta x \Delta y$, which lie wholly or partly in the area bounded by C'. Through these lines pass planes parallel to OZ. These planes will intersect the surface in curves which intersect in points the projections of which on the plane XOY are the vertices of the rectangles; for example, M is the projection of P. At every such point as P draw the tangent plane to the surface. From each tangent plane there will be cut a parallelogram * by the planes drawn parallel to OZ.

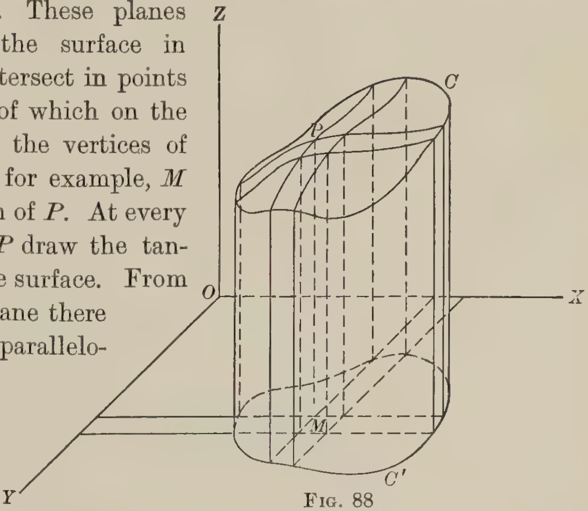

FIG. 88

We shall now define the area of the surface $f(x, y, z) = 0$, bounded by the curve C, as the limit of the sum of the areas of these parallelograms cut from the tangent planes, as their number is made to increase indefinitely, at the same time that $\Delta x \doteq 0$ and $\Delta y \doteq 0$. This definition involves the assumption that the limit is independent of the manner in which the tangent planes are drawn, or of the way in which the small areas are made to approach zero. This assumption may be proved by careful but somewhat intricate reasoning.

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* This parallelogram is not drawn in the figure, since it coincides so nearly with the surface element.

If ΔA denotes the area of one of these parallelograms in a tangent plane, and γ denotes the angle which the normal to the tangent plane makes with OZ, then (§ 92)

$$\Delta x \Delta y = \Delta A \cos \gamma, \tag{1}$$

since the projection of ΔA on the plane XOY is $\Delta x \Delta y$. The direction cosines of the normal are, by § 112, (2), proportional to $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, -1; hence

$$\cos \gamma = \frac{1}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}},$$

and hence

$$\Delta A = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \Delta x \Delta y, \tag{2}$$

and

$$\sum \Delta A = \sum \sum \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \Delta x \Delta y. \tag{3}$$

According to the definition, to find A we must take the limit of (3) as $\Delta x \doteq 0$ and $\Delta y \doteq 0$; that is

$$A = \iint \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dx dy, \tag{4}$$

where the integration must be extended over the area in the plane XOY bounded by the curve C'.

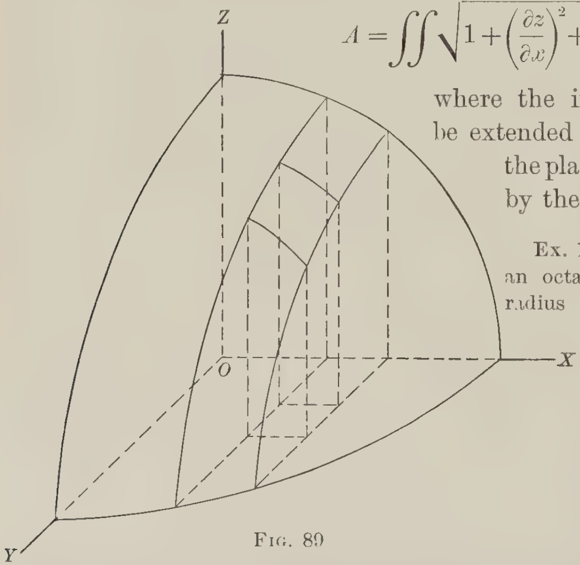

Fig. 89

Ex. 1. Find the area of an octant of a sphere of radius a.

If the center of the sphere is taken as the origin of coordinates (fig. 89), the equation of the sphere is

$$x^2 + y^2 + z^2 = a^2, \tag{1}$$

and the projection of the required area on the plane XOY is the area in the first quadrant bounded by the circle

$$x^2 + y^2 = a^2 \tag{2}$$

and the axes OX and OY.

From (1), $$\frac{\partial z}{\partial x} = -\frac{x}{z}, \qquad \frac{\partial z}{\partial y} = -\frac{y}{z}.$$

$$\therefore \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{\frac{x^2 + y^2 + z^2}{z^2}} = \frac{a}{\sqrt{a^2 - x^2 - y^2}}. \qquad \text{by (1)}$$

$$\therefore A = \int_0^a \int_0^{\sqrt{a^2 - x^2}} \frac{a\,dx\,dy}{\sqrt{a^2 - x^2 - y^2}}$$

$$= \tfrac{1}{2}\pi a \int_0^a dx$$

$$= \tfrac{1}{2}\pi a^2.$$

EX. 2. The center of a sphere of radius $2a$ is on the surface of a right circular cylinder of radius a. Find the area of the part of the cylinder intercepted by the sphere.

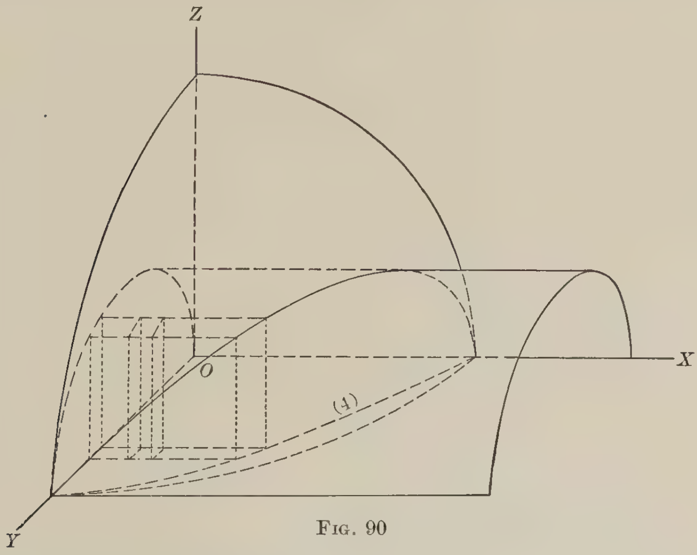

FIG. 90

Let the equation of the sphere be

$$x^2 + y^2 + z^2 = 4a^2, \qquad (1)$$

the center being at the origin (fig. 90), and let the equation of the cylinder be

$$y^2 + z^2 - 2ay = 0, \qquad (2)$$

the elements of the cylinder being parallel to OX.

To find the projection of the required area on the plane XOY it is necessary to find the equation of a cylinder passing through the line of intersection of (1) and (2), and having its elements parallel to OZ. Now

$$k_1(x^2 + y^2 + z^2 - 4a^2) + k_2(y^2 + z^2 - 2ay) = 0 \qquad (3)$$

represents any surface through the line of intersection of (1) and (2), and hence it only remains to choose k_1 and k_2 so that (3) shall be independent of z. Accordingly let $k_1 = 1$ and $k_2 = -1$, and (3) becomes

$$x^2 + 2ay - 4a^2 = 0. \tag{4}$$

From (2), $$\frac{\partial z}{\partial x} = 0, \qquad \frac{\partial z}{\partial y} = \frac{a-y}{z}.$$

$$\therefore \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{\frac{z^2 + (a-y)^2}{z^2}} = \frac{a}{\sqrt{2ay - y^2}}. \qquad \text{by (2)}$$

$$\begin{aligned} \therefore A &= \int_0^{2a} \int_{-\sqrt{4a^2-2ay}}^{\sqrt{4a^2-2ay}} \frac{a\,dy\,dx}{\sqrt{2ay - y^2}} \\ &= 2a\sqrt{2a} \int_0^{2a} \frac{dy}{\sqrt{y}} \\ &= 8a^2. \end{aligned}$$

The limits of integration were determined from (4), which is the projection of the bounding line of the required area on the plane XOY.

As an equal area is intercepted on the negative side of the plane XOY, the above result must be multiplied by 2. Hence the required area is $16a^2$.

The evaluation of (4) may sometimes be simplified by transforming to polar coördinates in the plane XOY.

Ex. 3. Find the area of the sphere $x^2 + y^2 + z^2 = a^2$ included in a cylinder having its elements parallel to OZ and one loop of the curve $r = a \cos 2\theta$ (fig. 91) in the plane XOY as its directrix.

Proceeding as in Ex. 1, we find the integrand $\dfrac{a}{\sqrt{a^2 - x^2 - y^2}}$. Transforming this integrand to polar coördinates, we have (§ 129)

$$A = 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{a\cos 2\theta} \frac{ar\,d\theta\,dr}{\sqrt{a^2 - r^2}},$$

for the first integration with respect to r covers a sector extending from the origin to a point on the curve $r = a\cos 2\theta$, since the curve passes through the origin; and the final integration with respect to θ is from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$, since the loop chosen is bounded by the radii vectors $\theta = -\frac{\pi}{4}$ and $\theta = \frac{\pi}{4}$. The factor 2 before the integral is necessary because there is an equal amount of area on the negative side of the plane XOY.

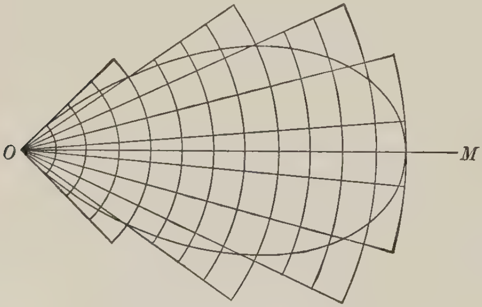

FIG. 91

Therefore
$$\begin{aligned} A &= 2a\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_0^{a\cos 2\theta}\frac{rd\theta dr}{\sqrt{a^2-r^2}} \\ &= 2a^2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(1-\sin 2\theta)\,d\theta \\ &= \pi a^2. \end{aligned}$$

If the required area is projected on the plane YOZ, we have

$$A = \iint\sqrt{1+\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2}\,dydz, \tag{5}$$

where the integration extends over the projection of the area on the plane YOZ; and if the required area is projected on the plane XOZ, we have

$$A = \iint\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dxdz, \tag{6}$$

where the integration extends over the projection of the area on the plane XOZ.

136. Center of gravity. In § 47 we defined the center of gravity of a system of particles all of which lie in the same plane, the resulting formulas being

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}, \qquad \bar{y} = \frac{\sum m_i y_i}{\sum m_i}.$$

If the particles do not all lie in the same plane, we are obliged to add a third equation,

$$\bar{z} = \frac{\sum m_i z_i}{\sum m_i},$$

to define the third coördinate of the center of gravity, the derivation of which is not essentially different from that given in § 47.

To determine the center of gravity of a physical body, we divide the body into elementary portions, the mass of any one of which may be represented by Δm. Then if (x_i, y_i, z_i) is any point such that the mass of one of the elementary portions may be considered as concentrated at that point, we define the center of gravity $(\bar{x}, \bar{y}, \bar{z})$ of the body by the formulas

$$\bar{x} = \text{Lim}\frac{\sum x_i\Delta m}{\sum \Delta m}, \quad \bar{y} = \text{Lim}\frac{\sum y_i\Delta m}{\sum \Delta m}, \quad z = \text{Lim}\frac{\sum z_i\Delta m}{\sum \Delta m}. \tag{1}$$

The denominator of each of the preceding fractions is evidently M, the mass of the body.

Formulas (1) can be expressed in terms of definite integrals, the evaluation of which gives the values of $\bar{x}$, $\bar{y}$, and $\bar{z}$. We shall here take up only those cases in which the definite integrals introduced are double or triple integrals.

137. Center of gravity of a plane area. The center of gravity of a plane area in the plane XOY has been defined in § 49. From that definition we have immediately that $\bar{z} = 0$.

To determine $\bar{x}$ and $\bar{y}$ we divide the area into rectangles of area $\Delta x \Delta y$ (fig. 86), and if we denote the density by ρ, $\Delta m = \rho \Delta x \Delta y$.

If we consider the mass of an element, as PQ, concentrated at P, we have, by substituting in (1), § 136,

$$\frac{\sum\sum x_i \rho \Delta x \Delta y}{\sum\sum \rho \Delta x \Delta y}, \tag{1}$$

an expression which is evidently less than $\bar{x}$; and if we consider the mass of PQ as concentrated at Q, we have

$$\frac{\sum\sum (x_i + \Delta x) \rho \Delta x \Delta y}{\sum\sum \rho \Delta x \Delta y} \tag{2}$$

an expression which is evidently greater than $\bar{x}$.

But the limits of (1) and (2) are the same (§ 3), for

$$\text{Lim} \frac{(x_i + \Delta x) \rho \Delta x \Delta y}{x_i \rho \Delta x \Delta y} = 1.$$

The limit of (1) is $\dfrac{\displaystyle\iint \rho x dx dy}{\displaystyle\iint \rho dx dy}$, both integrals being taken over the entire area.

$$\text{Therefore} \quad \bar{x} = \frac{\displaystyle\iint \rho x dx dy}{\displaystyle\iint \rho dx dy}, \qquad \bar{y} = \frac{\displaystyle\iint \rho y dx dy}{\displaystyle\iint \rho dx dy}, \tag{3}$$

$\bar{y}$ being derived in the same manner as $\bar{x}$.

If ρ is constant, it can be canceled; and in any problem in which ρ is not defined, it will be understood that it is constant. In that case the denominator of each coördinate is the area of the plane figure.

Ex. Find the center of gravity of the segment of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ cut off by the chord through the positive ends of the axes of the curve.

This is Ex. 2, § 49, and the student should compare the two solutions.

The equation of the chord is $bx + ay = ab$.

To determine $\bar{x}$ and $\bar{y}$ we have to compute the two integrals $\iint x dx dy$ and $\iint y dx dy$ over the shaded area of fig. 39, and also find that shaded area.

The area is the area of a quadrant of the ellipse less the area of the triangle formed by the coördinate axes and the chord, and is accordingly

$$\tfrac{1}{4}(\pi ab) - \tfrac{1}{2}ab = \tfrac{1}{4}ab(\pi - 2).$$

For the integrals the limits of integration with respect to y are $y_1 = \frac{ab - bx}{a}$ and $y_2 = \frac{b}{a}\sqrt{a^2 - x^2}$, y_1 being found from the equation of the chord, and y_2 being found from the equation of the ellipse. The limits for x are evidently 0 and a.

$$\begin{aligned}\int_0^a \int_{\frac{ab - bx}{a}}^{\frac{b}{a}\sqrt{a^2 - x^2}} x dx dy &= \int_0^a \left(\frac{b}{a} x\sqrt{a^2 - x^2} - bx + \frac{bx^2}{a}\right) dx \\ &= \tfrac{1}{6} ba^2.\end{aligned}$$

$$\begin{aligned}\int_0^a \int_{\frac{ab - bx}{a}}^{\frac{b}{a}\sqrt{a^2 - x^2}} y dx dy &= \frac{1}{a^2}\int_0^a (-b^2x^2 + ab^2x)\, dx \\ &= \tfrac{1}{6} b^2a.\end{aligned}$$

$$\therefore \bar{x} = \frac{2a}{3(\pi - 2)}, \qquad \bar{y} = \frac{2b}{3(\pi - 2)}.$$

138. If the equation of the bounding curve of the area is in polar coördinates, we have, by transforming equations (3), § 137, by § 129,

$$\begin{gathered}\bar{x} = \frac{\iint \rho r^2 \cos\theta dr d\theta}{\iint \rho r dr d\theta}, \\ \bar{y} = \frac{\iint \rho r^2 \sin\theta dr d\theta}{\iint \rho r dr d\theta}.\end{gathered} \tag{1}$$

Ex. Find the center of gravity of the area bounded by the two circles

$$r = a\cos\theta, \qquad r = b\cos\theta. \qquad (b > a)$$

It is evident from the symmetry of the area (fig. 92) that $\bar{y} = 0$.

As the denominator of the fractions, after canceling ρ, is the area, it is equal to $\frac{\pi b^2}{4} - \frac{\pi a^2}{4} = \frac{1}{4}\pi(b^2 - a^2)$.

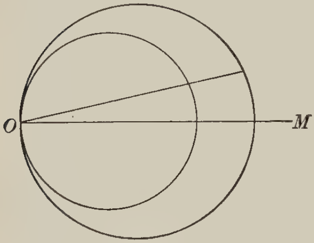

FIG. 92

The numerator for $\bar{x}$ becomes

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{a\cos\theta}^{b\cos\theta} r^2\cos\theta\, d\theta\, dr = \tfrac{1}{3}(b^3 - a^3)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\theta\, d\theta$$
$$= \tfrac{1}{8}\pi(b^3 - a^3).$$

$$\therefore\ \bar{x} = \frac{b^2 + ab + a^2}{2(b + a)}.$$

139. Center of gravity of a solid. To find the center of gravity of any solid we have merely to express the Δm of formulas (1), § 136, in terms of space coördinates and proceed as in § 137. For example, if rectangular coördinates are used, $\Delta m = \rho\Delta x\Delta y\Delta z$, and

$$\bar{x} = \frac{\iiint \rho x\, dx\, dy\, dz}{\iiint \rho\, dx\, dy\, dz},$$

$$\bar{y} = \frac{\iiint \rho y\, dx\, dy\, dz}{\iiint \rho\, dx\, dy\, dz}, \qquad (1)$$

$$\bar{z} = \frac{\iiint \rho z\, dx\, dy\, dz}{\iiint \rho\, dx\, dy\, dz},$$

the work of derivation being like that of § 137.

If desired, formulas (1) may be expressed in cylindrical or polar coördinates.

Ex. Find the center of gravity of a body of uniform density, bounded by one nappe of a right circular cone of vertical angle 2α and a sphere of radius a, the center of the sphere being at the vertex of the cone.

If the center of the sphere is taken as the origin of coördinates and the axis of the cone as the axis of z, it is evident from the symmetry of the solid that $\bar{x} = \bar{y} = 0$. To find $\bar{z}$, we shall use polar coördinates, the equations of the sphere and the cone being respectively $r = a$ and $\phi = \alpha$.

Then
$$\bar{z} = \frac{\int_0^{2\pi}\int_0^{\alpha}\int_0^{a} r\cos\phi\cdot r^2\sin\phi\, d\theta\, d\phi\, dr}{\int_0^{2\pi}\int_0^{\alpha}\int_0^{a} r^2\sin\phi\, d\theta\, d\phi\, dr}.$$

The denominator is the volume of a spherical cone the base of which is a zone of altitude $a(1-\cos\alpha)$; therefore its volume equals $\frac{2}{3}\pi a^3(1-\cos\alpha)$ (§ 128).

$$\begin{aligned}\int_0^{2\pi}\int_0^{\alpha}\int_0^{a} r^3\cos\phi\sin\phi\, d\theta\, d\phi\, dr &= \tfrac{1}{4}a^4\int_0^{2\pi}\int_0^{\alpha}\cos\phi\sin\phi\, d\theta\, d\phi\\ &= \tfrac{1}{8}a^4(1-\cos^2\alpha)\int_0^{2\pi}d\theta\\ &= \tfrac{1}{4}\pi a^4(1-\cos^2\alpha).\end{aligned}$$

$$\therefore\ \bar{z} = \tfrac{3}{8}(1+\cos\alpha)a.$$

140. Volume. In §§ 126, 128 we found expressions for the element of volume in rectangular, in cylindrical, and in polar coördinates. The volume of a solid bounded by any surfaces will be the limit of the sum of these elements as their number increases indefinitely while their magnitudes approach the limit zero. It will accordingly be expressed as a triple integral.

Ex. 1. Find the volume bounded by the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

From symmetry (fig. 93) it is evident that the required volume is eight times the volume in the first octant bounded by the surface and the coördinate planes.

In summing up the rectangular parallelopipeds $\Delta x\Delta y\Delta z$ to form a prism with edges parallel to OZ, the limits for z are 0 and $c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$, the latter being found from the equation of the ellipsoid.

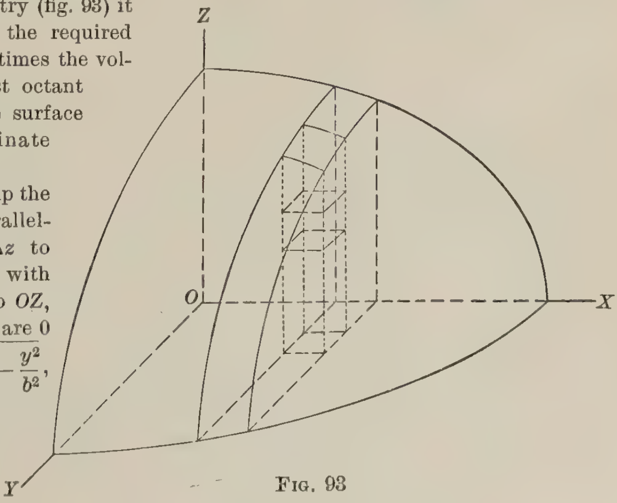

FIG. 93

Summing up next with respect to y, to obtain the volume of a slice, we have 0 as the lower limit of y, and $b\sqrt{1-\frac{x^2}{a^2}}$ as the upper limit. This latter limit is determined by solving the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, found by letting $z=0$ in the equation of the ellipsoid; for it is in the plane $z=0$ that the ellipsoid has the greatest extension in the direction OY, corresponding to any value of x.

Finally, the limits for x are evidently 0 and a.

Therefore
$$\begin{aligned} V &= 8\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dxdydz \\ &= 8c\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dxdy \\ &= 2\pi bc\int_0^a\left(1-\frac{x^2}{a^2}\right)dx \\ &= \tfrac{4}{3}\pi abc. \end{aligned}$$

It is to be noted that the first integration, when rectangular coördinates are used, leads to an integral of the form

$$\iint(z_2-z_1)\,dxdy,$$

where z_2 and z_1 are found from the equations of the bounding surfaces. It follows that many volumes may be found as easily by double as by triple integration.

In particular, if $z_1=0$, the volume is the one graphically representing the double integral (§ 121).

Ex. 2. Find the volume bounded by the surface $z=ae^{-(x^2+y^2)}$ and the plane $z=0$.

To determine this volume it will be advantageous to use cylindrical coördinates. Then the equation of the surface becomes $z=ae^{-r^2}$, and the element of volume is (§ 128) $rdrd\theta dz$.

Integrating first with respect to z, we have as the limits of integration 0 and ae^{-r^2}. If we integrate next with respect to r, the limits are 0 and ∞, for in the plane $z=0$, $r=\infty$, and as z increases the value of r decreases toward zero as a limit. For the final integration with respect to θ the limits are 0 and 2π.

Therefore
$$\begin{aligned} V &= \int_0^{2\pi}\int_0^{\infty}\int_0^{ae^{-r^2}} rd\theta drdz \\ &= a\int_0^{2\pi}\int_0^{\infty} re^{-r^2}d\theta dr \\ &= \tfrac{1}{2}a\int_0^{2\pi}d\theta \\ &= \pi a. \end{aligned}$$

In the same way that the computation of the volume in Ex. 2 has been simplified by the use of cylindrical coördinates, the computation of a volume may be simplified by a change to polar coördinates; and the student should always keep in mind the possible advantage of such a change.

141. Moment of inertia of a solid. The moment of inertia of a solid about an axis may be found as follows: Divide the solid into elements of volume, and let Δm represent the mass of such an element. Let h and $h+\Delta h$ represent the least and the greatest distances of any particle of Δm from the axis. Then if Δm is regarded as concentrated at the least distance, its moment of inertia would be $h^2\Delta m$; and if Δm is regarded as concentrated at the greatest distance, its moment of inertia would be $(h+\Delta h)^2\Delta m$. If the moment of inertia of the entire solid is denoted by M,

$$\sum h^2\Delta m < M < \sum (h+\Delta h)^2\Delta m,$$

where the two sums include all the elements of volume into which the solid was divided.

$\text{Lim}\,\dfrac{(h+\Delta h)^2\Delta m}{h^2\Delta m}=1$, when the number of the elements of volume increases indefinitely while their magnitude approaches the limit zero.

Hence we define M by the equation

$$M = \text{Lim}\sum h^2\Delta m. \tag{1}$$

It is to be noted that the cases of §§ 130–132 are but special cases of (1).

The computation of M requires us to express (1) as a definite integral in terms of some system of coördinates, the choice of a particular system of coördinates depending upon the solid.

Ex. Find the moment of inertia of a homogeneous sphere of density ρ and radius a about a diameter.

We shall take the center of the sphere as the origin of coördinates, and the diameter about which the moment is to be taken as the axis of z. The problem will then be most easily solved by using cylindrical coördinates.

The equation of the sphere will be $r^2 + z^2 = a^2$, and $dm = \rho r dr d\theta dz$, where ρ is the density; also $h = r$, so that we have to find the value of the triple integral $\rho\int\int\int r^3 d\theta dr dz$.

Integrating first with respect to z, we find the limits, from the equation of the sphere, to be $-\sqrt{a^2-r^2}$ and $\sqrt{a^2-r^2}$. Integrating next with respect to r, we have the limits 0 and a, thereby finding the moment of a sector of the sphere. To include all the sectors, we have to take 0 and 2π as the limits of θ in the last integration.

Therefore
$$M = \rho\int_0^{2\pi}\int_0^a\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r^3 d\theta dr dz.$$

As a result of the first integration,
$$M = 2\rho\int_0^{2\pi}\int_0^a r^3\sqrt{a^2-r^2}\,d\theta dr.$$

Making the next integration by a reduction formula or a trigonometric substitution, we have
$$M = \tfrac{4}{15}\rho a^5\int_0^{2\pi} d\theta = \tfrac{8}{15}\pi\rho a^5.$$

142. Attraction. In § 45 the attraction between two particles was defined, and the component in the direction OX of the attraction of any body on a particle was derived as $\lim\limits_{n=\infty}\sum\frac{\cos\theta_i}{r_i^2}\Delta m$, where Δm represents an element of mass of the body, r_i may be considered the shortest distance from any point of the element to the particle, and θ_i is the angle between OX and the line r_i. This expression is entirely general, and similar expressions may be derived for the components of the attraction in the directions OY and OZ.

Now that we can use double and triple integrals the application of these formulas is simplified.

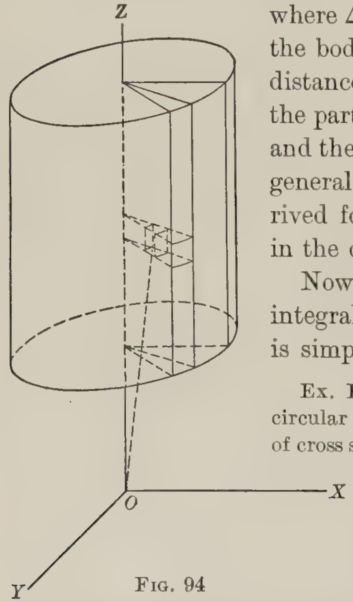

Fig. 94

Ex. Find the attraction due to a homogeneous circular cylinder of density ρ, of height h, and radius of cross section a, on a particle in the line of the axis of the cylinder at a distance b units from one end of the cylinder.

Take the particle at the origin of coördinates (fig. 94), and the axis of the cylinder as OZ. Using cylindrical coördinates, we have $dm = \rho r dr d\theta dz$ and $r_i = \sqrt{z^2+r^2}$. From the symmetry of the figure the resultant components of attraction in the directions OX and OY are zero, and $\cos\theta_i = \frac{z}{\sqrt{z^2+r^2}}$ for the resultant component in the direction OZ.

Therefore, letting A_z represent the component in the direction OZ, we have

$$A_z = \rho \int_0^{2\pi} \int_0^{a} \int_b^{b+h} \frac{rz}{(z^2 + r^2)^{\frac{3}{2}}}\, d\theta dr dz,$$

where the limits of integration are evident from fig. 94.

$$\begin{aligned} A_z &= \rho \int_0^{2\pi} \int_0^{a} \left(\frac{r}{\sqrt{b^2 + r^2}} - \frac{r}{\sqrt{(b + h)^2 + r^2}} \right) d\theta dr \\ &= \rho \int_0^{2\pi} \left(h + \sqrt{b^2 + a^2} - \sqrt{(b + h)^2 + a^2} \right) d\theta \\ &= 2\pi\rho \left(h + \sqrt{b^2 + a^2} - \sqrt{(b + h)^2 + a^2} \right). \end{aligned}$$

PROBLEMS

1. Find the moment of inertia of the area between the straight lines $x + y = 1$, $x = 1$, and $y = 1$ about an axis perpendicular to its plane at O.

2. Find the moment of inertia of the area bounded by the parabolas $y^2 = 4ax + 4a^2$, $y^2 = -4bx + 4b^2$ about an axis perpendicular to its plane at O.

3. Find the moment of inertia of the area of the loop of the curve $b^2y^2 = x^2(a - x)$ about an axis perpendicular to its plane at O.

4. Determine the moment of inertia about an axis perpendicular to the plane at the pole of the area included between the straight line $r = a \sec \theta$ and two straight lines at right angles to each other passing through the pole, one of these lines making an angle of $60°$ with the initial line.

5. Find the moment of inertia of the area of one loop of the curve $r = a \cos 3\theta$ about an axis perpendicular to its plane at the pole.

6. Find the moment of inertia of the area of the cardioid $r = a(\cos \theta + 1)$ about an axis perpendicular to its plane at the pole.

7. Find the moment of inertia of the area of one loop of the lemniscate $r^2 = 2a^2 \cos 2\theta$ about an axis perpendicular to its plane at the pole.

8. Find the moment of inertia of the total area bounded by the curve $r^2 = a^2 \sin \theta$ about an axis perpendicular to its plane at the pole.

9. Find the moment of inertia of the entire area bounded by the curve $r^2 = a^2 \sin 3\theta$ about an axis perpendicular to its plane at the pole.

10. Find the moment of inertia of the area of a circle of radius a about an axis perpendicular to its plane at any point on its circumference.

11. Find the moment of inertia of the area of the circle $r = a$ which is not included in the curve $r = a \sin 2\theta$ about an axis perpendicular to its plane at the pole.

12. Find the moment of inertia about the axis of y of the area bounded by the hyperbola $xy = a^2$ and the line $2x + 2y - 5a = 0$.

13. Find the moment of inertia about the axis of x of the area of the loop of the curve $b^2y^2 = x^2(a - x)$.

14. Find the moment of inertia of the area of one loop of the lemniscate $r^2 = 2a^2 \cos 2\theta$ about the straight line through the pole perpendicular to the initial line as an axis.

15. Find the moment of inertia of the area of the cardioid $r = a(\cos\theta + 1)$ above the initial line, about the initial line as an axis.

16. Find the moment of inertia of the area bounded by a semicircle of radius $2a$ and the corresponding diameter, about the tangent parallel to the diameter.

17. Find the area bounded by the hyperbola $xy = a^2$ and the line $2x + 2y - 5a = 0$.

18. Find the area bounded by the parabola $x^2 = 4ay$ and the witch $y = \frac{8a^3}{x^2 + 4a^2}$.

19. Find the area bounded by the limaçon $r = 2\cos\theta + 3$ and the circle $r = 2\cos\theta$.

20. Find the area bounded by the confocal parabolas $y^2 = 4ax + 4a^2$, $y^2 = -4bx + 4b^2$.

21. Find the area bounded by the circles $r = a\cos\theta$, $r = a\sin\theta$.

22. Find the areas of the three parts of the circle $x^2 + y^2 - 2ax = 0$ into which it is divided by the parabola $y^2 = ax$.

23. Find the area cut off from the lemniscate $r^2 = 2a^2\cos 2\theta$ by the straight line $r\cos\theta = \frac{\sqrt{3}}{2}a$.

24. Find the area of the surface cut from the cylinder $x^2 + y^2 = a^2$ by the cylinder $y^2 + z^2 = a^2$.

25. Find the area of the surface of a sphere of radius a intercepted by a right circular cylinder of radius $\frac{1}{2}a$, if an element of the cylinder passes through the center of the sphere.

26. Find the area of the surface of the cone $x^2 + y^2 - z^2 = 0$ cut out by the cylinder $x^2 + y^2 - 2ax = 0$.

27. Find the area of the surface of the cylinder $x^2 + y^2 - 2ax = 0$ bounded by the plane XOY and a right circular cone having its vertex at O, its axis along OZ, and its vertical angle equal to 90°.

28. Find the area of the surface of the right circular cylinder $z^2 + (x\cos\alpha + y\sin\alpha)^2 = a^2$ included in the first octant.

29. On the double ordinates of the circle $x^2 + y^2 = a^2$ as bases, and in planes perpendicular to the plane of the circle, isosceles triangles, each with vertical angle 2α, are described. Find the equation of the convex surface thus formed, and its total area.

30. Find the area of the surface $z = xy$ included in the cylinder $(x^2 + y^2)^2 = x^2 - y^2$.

31. Find the area of the sphere $x^2 + y^2 + z^2 = a^2$ included in the cylinder with elements parallel to OZ, and having for its directrix in the plane XOY a single loop of the curve $r = a\cos 3\theta$.

32. Find the area of that part of the surface $z = \frac{x^2 - y^2}{2a}$ the projection of which on the plane XOY is bounded by the curve $r^2 = a^2\cos\theta$.

33. Find the area of the sphere $x^2 + y^2 + z^2 = 4\,a^2$ bounded by the intersection of the sphere and the right cylinder, the elements of which are parallel to OZ and the directrix of which is the cardioid $r = a\,(\cos\theta + 1)$ in the plane XOY.

34. Find the center of gravity of the plane area bounded by the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ and the line $x + y = a$.

35. Find the center of gravity of the plane area bounded by the parabola $x^2 = 4\,ay$ and the witch $y = \dfrac{8\,a^3}{x^2 + 4\,a^2}$.

36. Find the center of gravity of the plane area bounded by the cissoid $y^2 = \dfrac{x^3}{2\,a - x}$ and its asymptote.

37. Find the center of gravity of the area of the part of the loop of the curve $a^4y^2 = a^2x^4 - x^6$ which lies in the first quadrant.

38. Find the center of gravity of the area in the first quadrant bounded by the curves $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ and $x^2 + y^2 = a^2$.

39. Find the center of gravity of the plane area bounded by OX, OY, and the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

40. A plate is in the form of a sector of a circle of radius a, the angle of the sector being $2\,\alpha$. If the thickness varies directly as the distance from the center, find its center of gravity.

41. How far from the origin is the center of gravity of the area included in a loop of the curve $r = a\cos 2\,\theta$?

42. Find the center of gravity of the area bounded by the cardioid $r = a\,(\cos\theta + 1)$.

43. Find the center of gravity of a thin plate of uniform thickness and density in the form of a loop of the lemniscate $r^2 = 2\,a^2\cos 2\,\theta$.

44. Find the center of gravity of a homogeneous body in the form of an octant of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

45. Find the center of gravity of the homogeneous solid bounded by the surfaces $z = k_1x$, $z = k_2x\,(k_2 > k_1)$, $x^2 + y^2 = 2\,ax$.

46. The density of a solid bounded by the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ varies directly as the distance from the plane YOZ. Find the center of gravity of the portion of this solid lying in the first octant.

47. Find the center of gravity of the homogeneous solid bounded by the surfaces $z = 0$, $y = 0$, $y = b$, $b^2z^2 = y^2\,(a^2 - x^2)$.

48. A homogeneous solid is bounded by a sphere of radius a and a right circular cone, the vertical angle of which is $\dfrac{\pi}{3}$, the vertex of which is on the surface of the sphere, and the axis of which coincides with a diameter of the sphere. Find its center of gravity.

49. Find the center of gravity of a right circular cone of altitude a, the density of each circular slice of which varies as the square of its distance from the vertex.

50. Find the center of gravity of an octant of a sphere of radius a, if the density varies as the distance from the center of the sphere.

51. Find the center of gravity of a homogeneous solid bounded by the surfaces of a right circular cone and a hemisphere of radius a, which have the same base and the same vertex.

52. Find the volume bounded by the surface $x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}} = a^{\frac{1}{2}}$ and the coördinate planes.

53. Find the volume of the part of the cylinder $x^2 + y^2 - 2\,ax = 0$ included between the planes $z = k_1x$, $z = k_2x\ (k_1 < k_2)$.

54. Find the total volume bounded by the surface $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^4}{c^4} = 1$.

55. Find the volume included in the first octant of space between the coördinate planes and the surface $y^2 - 16\left(1 - \dfrac{x}{3}\right)^2 + 8\,z\left(1 - \dfrac{x}{3}\right) = 0$.

56. Find the volume in the first octant bounded by the surfaces $z = (x + y)^2$, $x^2 + y^2 = a^2$.

57. Find the volume bounded by the surface $b^2z^2 = y^2(a^2 - x^2)$ and the planes $y = 0$ and $y = b$.

58. Find the volume bounded by the surfaces $az = xy$, $x + y + z = a$, $z = 0$.

59. Find the volume bounded by the cylindroid $z^2 = x + y$ and the planes $x = 0$, $y = 0$, $z = 2$.

60. Find the volume of the paraboloid $y^2 + z^2 = 8\,x$ cut off by the plane $y = 2\,x - 2$.

61. Find the volume bounded by the surfaces $z = ax^2 + by^2$, $y^2 = 2\,cx - x^2$, $z = 0$.

62. Find the volume bounded by the surfaces $x^2 + y^2 = ax$, $x^2 + y^2 = bz$, $z = 0$.

63. Find the total volume bounded by the surface $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}$.

64. Find the volume cut from a sphere of radius a by a right circular cylinder of radius $\dfrac{a}{2}$, one element of the cylinder passing through the center of the sphere.

65. Find the volume bounded by the surfaces $z = a(x + y)$, $z = a(x^2 + y^2)$.

66. Find the volume bounded by the surfaces $z = 0$, $z = ar^2$, $r = b\cos\theta$.

67. Find the total volume bounded by the surface $(x^2 + y^2 + z^2)^3 = 27\,a^3xyz$. (Change to polar coördinates.)

68. Find the volume bounded by a sphere of radius a and a right circular cone, the axis of the cone coinciding with a diameter of the sphere, the vertex being at one end of the diameter and the vertical angle of the cone being 60°.

69. Find the total volume bounded by the surface $(x^2 + y^2 + z^2)^2 = axyz$.

70. Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$ included in a cylinder with elements parallel to OZ, and having for its directrix in the plane XOY one loop of the curve $r = a\cos 3\,\theta$.

71. Find the volume bounded by the plane XOY, the cylinder $x^2 + y^2 - 2ax = 0$, and the right circular cone having its vertex at O, its axis coincident with OZ, and its vertical angle equal to 90°.

72. Find the total volume bounded by the surface $r^2 + z^2 = ar(\cos\theta + 1)$.

73. Find the moment of inertia of a homogeneous ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, of density ρ, about OX.

74. Find the moment of inertia about its axis of a homogeneous right circular cylinder of density ρ, height h, and radius a.

75. A solid is in the form of a right circular cone of altitude h and vertical angle 2α. Find its moment of inertia about its axis, if the density of any particle is proportional to its distance from the base of the cone.

76. The density of a solid sphere of radius a varies as the distance from a diametral plane. Find its moment of inertia about the diameter perpendicular to the above diametral plane.

77. A homogeneous solid of density ρ is in the form of a hemispherical shell, the inner and the outer radii of which are respectively r_1 and r_2. Find its moment of inertia about any diameter of the base of the shell.

78. A solid is bounded by the plane $z = 0$, the cone $z = r$ (cylindrical coördinates), and the cylinder having its elements parallel to OZ and its directrix one loop of the lemniscate $r^2 = 2a^2 \cos 2\theta$ in the plane XOY. Find its moment of inertia about OZ, if the density varies as the distance from OZ.

79. Find the attraction of a homogeneous right circular cone of mass M, altitude h, and vertical angle 2α on a particle at its vertex.

80. A portion of a right circular cylinder of radius a and uniform density ρ is bounded by a spherical surface of radius $b\,(b > a)$, the center of which coincides with the center of the base of the cylinder. Find the attraction of this portion of the cylinder on a particle at the middle point of its base.

81. Find the attraction due to a hemisphere of radius a on a particle at the center of its base, if the density varies directly as the distance from the base.

82. The density of a hemisphere of radius a varies directly as the distance from the base. Find its attraction on a particle in the straight line perpendicular to the base at its center, and at the distance a from the base in the direction away from the hemisphere.

83. A homogeneous ring is bounded by the plane XOY, a sphere of radius $2a$ with center at O, and a right circular cylinder of radius a, the axis of which coincides with OZ. Find the attraction of the ring on a particle at O.

CHAPTER XIV

LINE INTEGRALS AND EXACT DIFFERENTIALS

143. Definition. Let C (fig. 95) be any curve in the plane XOY connecting the two points L and K, and let M and N be two functions of x and y which are one-valued and continuous for all points on C. Let C be divided into n segments by the points $P_1, P_2, P_3, \cdots, P_{n-1}$, and let Δx be the projection of one of these segments on OX and Δy its projection on OY. That is, $\Delta x = x_{i+1} - x_i$, $\Delta y = y_{i+1} - y_i$, where the values of Δx and Δy are not necessarily the same for all values of i. Let the value of M for each of the n points $L, P_1, P_2, \cdots, P_{n-1}$ be multiplied by the corresponding value of Δx, and the value of N for the same point by the corresponding value of Δy, and let the sum be formed

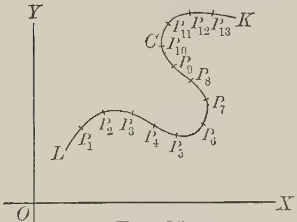

FIG. 95

$$\sum_{i=0}^{i=n-1}[M(x_i, y_i)\Delta x + N(x_i, y_i)\Delta y].$$

The limit of this sum as n increases without limit and Δx and Δy approach zero as a limit is denoted by

$$\int_{(C)}(Mdx + Ndy),$$

and is called a *line integral along the curve C*. The point K may coincide with the point L, thus making C a closed curve.

If x and y are expressed in terms of a single independent variable from the equation of the curve, the line integral reduces to a definite integral of the ordinary type; but this reduction is not always convenient or possible, and it is important to study the properties of the line integral directly.

We shall give first a few examples, showing the importance of the line integral in some practical problems.

Ex. 1. *Work.* Let us assume that at every point of the plane a force acts, which varies from point to point in magnitude and direction. We wish to find the work done on a particle moving from L to K along the curve C. Let C be divided into segments, each of which is denoted by Δs and one of which is represented in fig. 96 by PQ. Let F be the force acting at P, PR the direction in which it acts, PT the tangent to C at P, and θ the angle RPT. Then the component of F in the direction PT is $F\cos\theta$, and the work done on a particle moving from P to Q is $F\cos\theta\Delta s$, except for infinitesimals of higher order. The work done in moving the particle along C is, therefore,

$$W = \text{Lim}\sum F\cos\theta\Delta s = \int_{(C)} F\cos\theta ds.$$

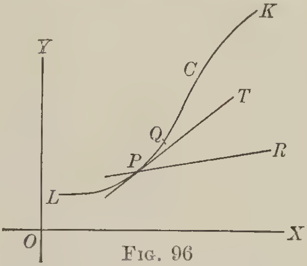

FIG. 96

Now let α be the angle between PR and OX, and ϕ the angle between PT and OX. Then $\theta = \phi - \alpha$ and $\cos\theta = \cos\phi\cos\alpha + \sin\phi\sin\alpha$. Therefore

$$W = \int_{(C)} (F\cos\phi\cos\alpha + F\sin\phi\sin\alpha)\, ds.$$

But $F\cos\alpha$ is the component of force parallel to OX and is usually denoted by X. Also $F\sin\alpha$ is the component of force parallel to OY and is usually denoted by Y. Moreover $\cos\phi ds = dx$ and $\sin\phi ds = dy$ (§ 42). Hence we have finally

$$W = \int_{(C)} (Xdx + Ydy).$$

Ex. 2. *Flow of a liquid.* Suppose a liquid flowing over a plane surface, the lines in which the particles flow being indicated by the curved arrows of fig. 97. We imagine the flow to take place in planes parallel to XOY, and shall assume the depth of the liquid to be unity. We wish to find the amount of liquid per unit of time which flows across a curve C.

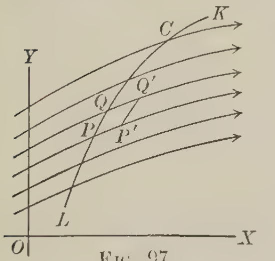

FIG. 97

Let q be the velocity of the liquid, α the angle which the direction of its motion at each point makes with OX, $u = q\cos\alpha$ the component of velocity parallel to OX, and $v = q\sin\alpha$ the component of velocity parallel to OY. Take an element of the curve $PQ = \Delta s$. In the time dt the particles of liquid which are originally on PQ will flow to $P'Q'$, where $PP' = QQ' = qdt$ (except for infinitesimals of higher order). The amount of liquid crossing PQ is therefore the amount in a cylinder with base $PQQ'P'$ and altitude unity. The volume of this cylinder is $PP'\cdot PQ\sin\theta = qdt\sin\theta\Delta s$, where $\theta = P'PQ$. Hence the amount of liquid crossing the whole curve C in the time dt is

$$\text{Lim}\sum qdt\sin\theta\Delta s = dt\,\text{Lim}\sum q\sin\theta\Delta s = dt\int_{(C)} q\sin\theta ds$$

and the amount per unit of time is

$$\int_{(C)} q\sin\theta ds.$$

To put this in the standard form, let ϕ be the angle made by PQ with OX. Then $\theta = \phi - \alpha$ and $\sin\theta = \sin\phi\cos\alpha - \cos\phi\sin\alpha$. Hence

$$q\sin\theta ds = q\sin\phi\cos\alpha ds - q\cos\phi\sin\alpha ds = -vdx + udy.$$

Therefore the amount of liquid flowing across C per unit of time is

$$\int_{(C)}(-vdx + udy).$$

Ex. 3. *Heat.* Consider a substance in a given state of pressure p, volume v, and temperature t. Then p, v, t are connected by a relation $f(p, v, t) = 0$, so that any two of them may be taken as independent variables. For a perfect gas $pv = kt$, if t is the absolute temperature, and the state of the substance is indicated by a point on the surface of fig. 55, or equally well by a point on any one of the three coördinate planes, since a point on the surface is uniquely determined by a point on one of these planes. We shall take t and v as the independent variables and shall therefore work on the (t, v) plane.

Now if Q is the amount of heat in the substance and an amount dQ is added, there result changes dp, dv, dt in p, v, and t respectively, and, except for infinitesimals of higher order,

$$dQ = Adp + Bdv + Cdt.$$

From the fundamental relation $f(p, v, t) = 0$, it follows that

$$\frac{\partial f}{\partial p}dp + \frac{\partial f}{\partial v}dv + \frac{\partial f}{\partial t}dt = 0,$$

whence we have

$$dQ = Mdt + Ndv.$$

Hence the total amount of heat introduced into the substance by a variation of its state indicated by the curve C is

$$Q = \int_{(C)}(Mdt + Ndv).$$

Ex. 4. *Area.* Consider a closed curve C (fig. 98) tangent to the straight lines $x = a$, $x = b$, $y = d$, and $y = e$, and of such shape that a straight line parallel to either of the coördinate axes intersects it in not more than two points. Let the ordinate through any point M intersect C in P_1 and P_2, where $MP_1 = y_1$ and $MP_2 = y_2$. Then, if A is the area inclosed by the curve,

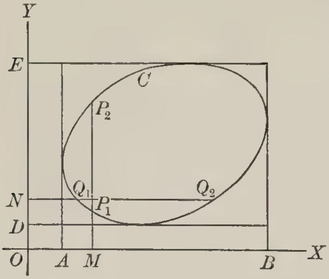

FIG. 98

$$\begin{aligned} A &= \int_a^b y_2dx - \int_a^b y_1dx \\ &= -\int_b^a y_2dx - \int_a^b y_1dx \\ &= -\int_{(C)} ydx, \end{aligned}$$

the last integral being taken around C in a direction opposite to the motion of the hands of a clock.

Similarly, if the line NQ_2 intersects C in Q_1 and Q_2, where $NQ_1 = x_1$ and $NQ_2 = x_2$, we have

$$A = \int_d^e x_2 dy - \int_d^e x_1 dy$$
$$= \int_d^e x_2 dy + \int_e^d x_1 dy$$
$$= \int_{(C)} x dy,$$

the last integral being taken also in the direction opposite to the motion of the hands of a clock. By adding the two values of A we have

$$2A = \int_{(C)} (-y dx + x dy).$$

If we apply this to find the area of an ellipse, we may take $x = a\cos\phi$, $y = b\sin\phi$ (I, § 166). Then $A = \frac{1}{2}\int_0^{2\pi} ab d\phi = \pi ab.$

144. Fundamental theorem. In using integrals around closed curves, we need some means of distinguishing between the two directions in which the curve may be traversed. Accordingly, when the curve is a portion of the boundary of a specified area, we shall define the positive direction around the curve as that in which a person should walk in order to keep the area on his left hand. Thus in fig. 99, where the shaded area is bounded by two curves, the positive direction of each curve is indicated by the arrows.

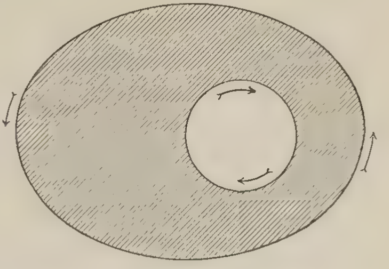

FIG. 99

With this convention, the fundamental theorem in the use of line integrals is as follows:

If M, N, $\frac{\partial M}{\partial y}$, and $\frac{\partial N}{\partial x}$ are continuous and one-valued in a closed area A and on its boundary curve C, then

$$\int\int_{(A)} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) dx dy = -\int_{(C)} (M dx + N dy),$$

where the double integral is taken over A and the line integral is taken in the positive direction around C.

To prove this, we shall first assume that the closed area is of the form described in Ex. 4, § 143, and shall use the notation of that example and fig. 98. Then

$$\begin{aligned}\iint_{(A)} \frac{\partial M}{\partial y}\,dxdy &= \int_a^b dx \int_{y_1}^{y_2} \frac{\partial M}{\partial y}\,dy\\ &= \int_a^b [M(x, y_2) - M(x, y_1)]\,dx\\ &= \int_a^b M(x, y_2)\,dx - \int_a^b M(x, y_1)\,dx\\ &= -\int_b^a M(x, y_2)\,dx - \int_a^b M(x, y_1)\,dx\\ &= -\int_{(C)} Mdx, \qquad (1)\end{aligned}$$

the last integral being taken in the positive direction around C. Similarly,

$$\iint_{(A)} \frac{\partial N}{\partial x}\,dxdy = \int_{(C)} Ndy. \qquad (2)$$

By subtracting (2) from (1) we have the theorem proved for a closed curve of the simple type considered.

The theorem is now readily extended to any area which can be cut up into areas of this simple type. For example, consider the area bounded by the curve C (fig. 100). By drawing the straight line LK the area is divided into three areas A', A'', A''', and the theorem applied to each of these areas. By adding the three equations obtained we have on the left-hand side of the new equation the double integral over the area bounded by C, and on the right-hand side the line integral along C and the straight line LK traversed twice in opposite directions. The integrals along the line LK therefore cancel, leaving only the integral around C.

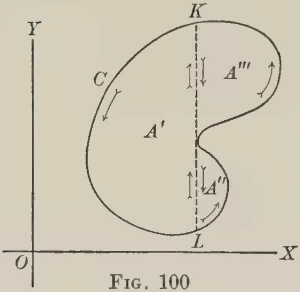

FIG. 100

The theorem is also true for areas bounded by more than one curve. Consider, for example, the area bounded by two curves C and C' (fig. 101). By drawing the line LK the area is turned into one bounded by a single curve, and the theorem may be applied to it. It appears that the line integral is taken twice along LK in opposite directions, and these two integrals cancel each other. The result is that the double integral over the area is equal to the line integral around each of the boundary curves in the positive direction of each. In the same way the theorem is shown to hold for areas bounded by any number of curves.

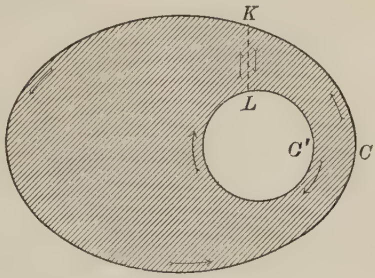

FIG. 101

Ex. If $M = y$ and $N = -x$, we have

$$\iint 2\,dxdy = -\int (ydx - xdy),$$

agreeing with the result of Ex. 4, § 143.

145. Line integrals of the first kind, $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. *If M and N are two functions of x and y, such that $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, and the discussion is restricted to a portion of the (x, y) plane in which M, N, $\dfrac{\partial M}{\partial y}$, and $\dfrac{\partial N}{\partial x}$ are continuous and one-valued, then*

(1) *The line integral $\int (Mdx + Ndy)$ around any closed curve is zero.*

(2) *The line integral $\int (Mdx + Ndy)$ between any two points is a function of the coördinates of the points, and is independent of the curve connecting the points.*

(3) *There exists a function $\phi(x, y)$ such that $\dfrac{\partial \phi}{\partial x} = M$, $\dfrac{\partial \phi}{\partial y} = N$.*

Conversely, if any one of the conclusions (1), (2), *or* (3) *is fulfilled, then $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.*

To prove (1), consider the fundamental theorem of § 144. It is at once evident that if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, $\int (Mdx + Ndy) = 0$ around any closed curve.

To prove (2), let L and K be any two points, and C and C' (fig. 102) any two curves connecting them. Let I be the value of the line integral from L to K along C, and I' the value of the line integral from L to K along C'. Then $-I'$ is the value of the line integral from K to L along C' Now by (1)

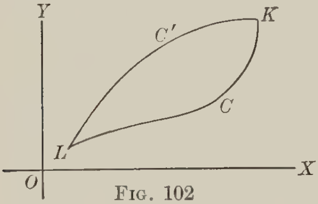

FIG. 102

$$I + (-I') = 0.$$

Therefore $$I = I'.$$

To prove (3), consider the line integral $$\int_{(x_0,\, y_0)}^{(x,\, y)} (Mdx + Ndy),$$

where (x_0, y_0) is a fixed point and (x, y) a variable point. By (2) this integral is independent of the path and is therefore fully determined when (x, y) is given. Hence, by the definition of a function of two variables, we may write

$$\int_{(x_0,\, y_0)}^{(x,\, y)} (Mdx + Ndy) = \phi(x, y).$$

Then $$\phi(x + h, y) = \int_{(x_0,\, y_0)}^{(x+h,\, y)} (Mdx + Ndy),$$

and since this integral is independent of the curve connecting the upper and the lower limits, we may take that curve as drawn first to (x, y) and then along a straight line to $(x + h, y)$. Then

$$\begin{aligned}\phi(x + h, y) &= \int_{(x_0,\, y_0)}^{(x,\, y)} (Mdx + Ndy) + \int_{(x,\, y)}^{(x+h,\, y)} (Mdx + Ndy) \\ &= \phi(x, y) + \int_{x}^{x+h} M(x, y)\, dx,\end{aligned}$$

since in the last integral y is constant and $dy = 0$.

Then, by § 30,

$$\phi(x + h, y) - \phi(x, y) = hM(\xi, y), \qquad (x < \xi < x + h)$$

whence $$\frac{\phi(x + h, y) - \phi(x, y)}{h} = M(\xi, y).$$

Letting h approach zero and taking the limit, we have

$$\frac{\partial \phi}{\partial x} = M(x, y).$$

In like manner we may show that

$$\frac{\partial \phi}{\partial y} = N(x, y),$$

and the third conclusion of the theorem is proved. It is to be noted that ϕ is determined except for an arbitrary constant which may be added. This constant depends upon the choice of the fixed point L. If another point L' is chosen, the value of ϕ differs from that obtained in using L by the value of the line integral between L and L'.

We must now show that, conversely, if any one of the conclusions of the theorem is fulfilled, then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Let us first assume that the integral around any closed path whatever is zero; we wish to show that $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0$. If $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ is not zero at all points, let us suppose it is not zero at a particular point K. Then, since $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous functions by hypothesis, $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ is also a continuous function, and hence has, at points sufficiently near to K, the same sign which it has at K. It is therefore possible to draw a closed curve around K, so that $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ has the same sign for all points in the area bounded by the curve. Then $\iint \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) dxdy$ taken over this area is not zero, and therefore $\int (Mdx + Ndy)$ taken around the bounding curve is not zero. But this is contrary to the hypothesis that $\int (Mdx + Ndy)$ taken around any curve whatever is zero. Hence $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ at all points.

Let us now assume that $\int (Mdx + Ndy)$ between any two points is independent of the curve connecting the points. This is equivalent to assuming that the integral around a closed curve is zero, and hence, as already shown, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Finally, let us assume that a function ϕ exists such that $\frac{\partial \phi}{\partial x} = M$, $\frac{\partial \phi}{\partial y} = N$. Then it follows that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, since each is equal to $\frac{\partial^2 \phi}{\partial x \partial y}$.

The theorem of this article is now completely proved.

Ex. 1. Let $M = \frac{-y}{x^2+y^2}$, $N = \frac{x}{x^2+y^2}$; then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{y^2 - x^2}{(x^2+y^2)^2}$. Hence $\int \frac{-y dx + x dy}{x^2+y^2} = 0$, if taken around a closed curve within which M, N, and their derivatives are finite and continuous. These conditions are met if the curve does not inclose the origin. In fact, if we introduce polar coördinates, placing $x = r\cos\theta$, $y = r\sin\theta$, then $\int \frac{-y dx + x dy}{x^2+y^2} = \int d\theta$. Now, for any path which does not inclose the origin, θ varies from its initial value α back to the same value, and therefore $\int d\theta = 0$. If the path winds once around the origin, θ varies from α to $\alpha + 2\pi$, and therefore $\int d\theta = 2\pi$.

The function ϕ of the general theorem is, in this example, equal to $\theta = \tan^{-1}\frac{y}{x}$.

Ex. 2. *Work.* If X and Y are components of force in a field of force, and $\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}$, then the work done on a particle moving around a closed curve is zero, and the work done in moving a particle between two points is independent of the path along which it is moved. Also there exists a function ϕ, called a force function, the derivatives of which with respect to x and y give the components of force parallel to the axes of x and y. It follows that the derivative of ϕ in any direction gives the force in that direction (Ex., § 111). Such a force as this is called a conservative force. Examples are the force of gravity and forces which are a function of the distance from a fixed point and directed along straight lines passing through that point.

Ex. 3. *Flow of a liquid.* If we consider a liquid flowing as in Ex. 2, § 143, it is clear that the net amount which flows over a closed curve is zero, since as much must leave the closed area as enters, there being no points within the area at which liquid is given out or drawn off. Hence $\int(-v dx + u dy)$, taken over any closed curve, is zero; and consequently $\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} = 0$. There exists also a function, usually denoted by ψ, such that $\frac{\partial \psi}{\partial x} = -v$, $\frac{\partial \psi}{\partial y} = u$.

146. Line integrals of the second kind, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. *In case M and N are such functions that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the value of the line integral $\int (Mdx + Ndy)$ depends upon the path, and there exists no function of which M and N are partial derivatives.*

Ex. 1. $\int_{(0,\,0)}^{(x_1,\,y_1)} (ydx - xdy)$.

Let us first integrate along a straight line connecting O and P_1 (fig. 103). The equation of the line is $y = \frac{y_1}{x_1}x$, and therefore along this line $ydx - xdy = 0$, and hence the value of the integral is zero.

Next, let us integrate along a parabola connecting O and P_1, the equation of which is $y^2 = \frac{y_1^2}{x_1}x$. Along this parabola

$$\int_{(0,\,0)}^{(x_1,\,y_1)} (ydx - xdy) = \frac{y_1}{2\sqrt{x_1}} \int_0^{x_1} \sqrt{x}\,dx = \tfrac{1}{3}\,x_1 y_1.$$

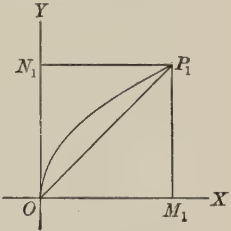

FIG. 103

Next, let us integrate along a path consisting of the two straight lines OM_1 and M_1P_1. Along OM_1, $y = 0$ and $dy = 0$; and along M_1P_1, $x = x_1$ and $dx = 0$. Hence the line integral reduces to $-\int_0^{y_1} x_1 dy = -x_1 y_1$.

Finally, let us integrate along a path consisting of the straight lines ON_1 and N_1P_1. Along ON_1, $x = 0$ and $dx = 0$; and along N_1P_1, $y = y_1$ and $dy = 0$. Therefore the line integral reduces to $\int_0^{x_1} y_1 dx = x_1 y_1$.

Ex. 2. *Work.* If the components of force X and Y in a field of force are such that $\frac{\partial X}{\partial y} \neq \frac{\partial Y}{\partial x}$, then the work done on a particle moving between two points depends upon the path of the particle, the work done on a particle moving around a closed path is not zero, and there exists no force function. Such a force is called a nonconservative force.

Ex. 3. *Heat.* If a substance is brought, by a series of changes of temperature, pressure, and volume, from an initial condition back to the same condition, the amount of heat acquired or lost by the substance is the mechanical equivalent of the work done, and is not in general zero. Hence the line integral $Q = \int (Mdt + Ndv)$ around a closed curve is not zero, and there exists no function whose partial derivatives are M and N. In fact, the heat Q is not a function of t and v, not being determined when t and v are given.

Ex. 4. *Adiabatic lines.* The line integral $\int (Mdt + Ndv)$ of Ex. 3 is zero if taken along a curve whose differential equation is

$$Mdt + Ndv = 0.$$

We will change this equation by replacing the variable t by p, which can be done by means of the fundamental relation $f(p, v, t) = 0$.

It is shown in the theory of heat that for a perfect gas the equation then becomes

$$vdp + \gamma pdv = 0,$$

where γ is a constant. The solution of this equation is

$$pv^{\gamma} = c.$$

If, then, a gas expands or contracts so as to obey this law, no heat is added to or subtracted from it. Such an expansion is called adiabatic expansion, and the corresponding curve is called an adiabatic curve.

The mathematical interest here is in the concrete illustration of a line integral being zero along any portion of a certain curve or family of curves.

In the line integrals of the kind before us, $\left(\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}\right)$, there is still a meaning to be attached to M and N. For, suppose the integral to be taken along the straight line $y = c$ from x to $x + \Delta x$, and let Δu be the value of the integral; that is,

$$\begin{aligned}\Delta u &= \int_{(x, y)}^{(x+\Delta x, y)} (Mdx + Ndy)\\ &= \int_x^{x+\Delta x} M(x, y)\, dx\\ &= \Delta x M(\xi, y), \qquad (\S\ 30)\end{aligned}$$

where $x < \xi < x + \Delta x$.

Consequently $$\lim_{\Delta x \doteq 0} \frac{\Delta u}{\Delta x} = M(x, y).$$

Similarly, $$\lim_{\Delta y \doteq 0} \frac{\Delta u}{\Delta y} = N(x, y).$$

We shall write these two equations as

$$\left(\frac{du}{dx}\right)_y = M, \qquad \left(\frac{du}{dy}\right)_x = N.$$

It is to be noticed that these derivatives are of different character from the partial derivatives of Chap. XI, since u is not a function of x and y. The property proved in § 117 does not hold here.

Ex. 5. *Heat.* Returning to the notation of Ex. 3, we have

$$M=\left(\frac{dQ}{dt}\right)_v, \qquad N=\left(\frac{dQ}{dv}\right)_t.$$

M is the limit of the ratio of the increment of heat to the increment of temperature when the volume is constant. Therefore, if we consider M as sensibly constant while the temperature changes by unity, M may be described as the amount of heat necessary to raise the temperature by one unit when the volume is constant.

Similarly, N may be described as the amount of heat necessary to change the volume by one unit when the temperature is constant.

147. Exact differentials. We have seen that if M and N are two functions of x and y satisfying the condition $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, there exists a function ϕ such that $\frac{\partial \phi}{\partial x}=M$, $\frac{\partial \phi}{\partial y}=N$. Then

$$Mdx+Ndy=\frac{\partial \phi}{\partial x}dx+\frac{\partial \phi}{\partial y}dy=d\phi,$$

and $Mdx+Ndy$ is called an *exact differential.*[*] Then also

$$\int_{(x_0,\, y_0)}^{(x_1,\, y_1)}(Mdx+Ndy)=\phi(x_1, y_1)-\phi(x_0, y_0),$$

the integral being independent of the curve connecting (x_0, y_0) and (x_1, y_1). The function ϕ may be found by computing the line integral along a conveniently chosen path, but it is usually more convenient to proceed as follows: Since $\frac{\partial \phi}{\partial x}=M(x, y)$, it follows that $\int M(x, y)\,dx$, where y is considered constant, will give that part of ϕ which contains x, but not necessarily all the part which contains y. We may write, therefore,

$$\phi(x, y)=\int Mdx+f(y),$$

where $f(y)$ is a function of y to be determined. This determination is made by using the relation $\frac{\partial \phi}{\partial y}=N$. Then $\frac{\partial}{\partial y}\int Mdx+f'(y)=N$, which is an equation from which $f(y)$ may be found.

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[*] In § 110, where the emphasis is on the fact that both x and y are varied, $d\phi$ is called a total differential. Here the emphasis is shifted to the fact that $d\phi$ is exactly obtained by the process of differentiation, and hence it is called an exact differential.

This method is valuable in solving the equation

$$Mdx + Ndy = 0,$$

when the condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is satisfied. Such an equation is called an *exact differential equation.* Its solution is

$$\phi = c,$$

where c is an arbitrary constant and ϕ is the function obtained in the manner just described.

Because of the importance of the method of obtaining ϕ, we give it in a rule as follows:

Integrate $\int Mdx$, *regarding* y *as constant and adding an unknown function of* y; *differentiate the result with respect to* y *and equate the new result to* N; *from the resulting equation determine the unknown function of* y.

If more convenient, the above rule may be replaced by the following:

Integrate $\int Ndy$, *regarding* x *as constant and adding an unknown function of* x; *differentiate the result with respect to* x *and equate the new result to* M; *from the resulting equation determine the unknown function of* x.

Ex. 1. $(4x^3 + 10xy^3 - 3y^4)dx + (15x^2y^2 - 12xy^3 + 5y^4)dy = 0.$

Here $\frac{\partial M}{\partial y} = 30xy^2 - 12y^3 = \frac{\partial N}{\partial x}$, and the equation is therefore exact. Proceeding according to the rule, we have

$$\begin{aligned}\phi &= \int (4x^3 + 10xy^3 - 3y^4)\,dx + f(y)\\ &= x^4 + 5x^2y^3 - 3xy^4 + f(y).\end{aligned}$$

Then from $\frac{\partial \phi}{\partial y} = N$, we have

$$15x^2y^2 - 12xy^3 + f'(y) = 15x^2y^2 - 12xy^3 + 5y^4,$$

whence $f'(y) = 5y^4$, and $f(y) = y^5$. The solution of the differential equation is therefore

$$x^4 + 5x^2y^3 - 3xy^4 + y^5 = c.$$

Ex. 2. $\left(\frac{1}{x}-\frac{y}{x\sqrt{y^2-x^2}}\right)dx+\frac{1}{\sqrt{y^2-x^2}}dy=0.$

Here $\frac{\partial M}{\partial y}=\frac{x}{(y^2-x^2)^{\frac{3}{2}}}=\frac{\partial N}{\partial x}$, and the equation is therefore exact. Following the second rule, we have

$$\phi=\int\frac{dy}{\sqrt{y^2-x^2}}=\log\left(y+\sqrt{y^2-x^2}\right)+f(x).$$

From $\frac{\partial \phi}{\partial x}=M$, we have

$$\frac{-x}{\sqrt{y^2-x^2}\left(y+\sqrt{y^2-x^2}\right)}+f'(x)=\frac{1}{x}-\frac{y}{x\sqrt{y^2-x^2}},$$

whence $f'(x)=0$, and $f(x)=c$. Hence the solution of the differential equation is

$$\log\left(y+\sqrt{y^2-x^2}\right)=c',$$

which reduces to $$x^2-2cy+c^2=0.$$

148. The integrating factor. We have seen that $Mdx+Ndy$ is not an exact differential when $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$. But in all cases there exist an infinite number of functions such that, if $Mdx+Ndy$ is multiplied by any one of them, it becomes an exact differential. Such functions are called *integrating factors.* That is, if μ is one of the integrating factors of $Mdx+Ndy$, then, by definition, $\mu(Mdx+Ndy)$ is an exact differential.

Ex. 1. The expression $ydx-xdy$ is not an exact differential. But

$$\frac{ydx-xdy}{x^2}=d\left(-\frac{y}{x}\right),$$

$$\frac{ydx-xdy}{x^2+y^2}=d\left(\tan^{-1}\frac{x}{y}\right),$$

$$\frac{ydx-xdy}{xy}=d\left(\log\frac{x}{y}\right)$$

are exact differentials, and the functions $\frac{1}{x^2}$, $\frac{1}{x^2+y^2}$, $\frac{1}{xy}$ are integrating factors.

To show that integrating factors always exist, we shall need to assume (§ 173) that the differential equation

$$Mdx+Ndy=0 \qquad (1)$$

has always a solution of the form

$$f(x,y,c)=0, \qquad (2)$$

where c is an arbitrary constant, and that (2) can be written

$$\phi(x,y)=c. \qquad (3)$$

Now (3) gives us
$$\frac{dy}{dx} = -\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}},$$

and (1) gives us
$$\frac{dy}{dx} = -\frac{M}{N};$$

and since (3) is the solution of (1), these two values of $\frac{dy}{dx}$ must be the same. That is,
$$\frac{\frac{\partial \phi}{\partial x}}{M} = \frac{\frac{\partial \phi}{\partial y}}{N} = \mu,$$

where μ is some function of x and y. Consequently

$$\mu(Mdx + Ndy) = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = d\phi,$$

and therefore μ is an integrating factor.

To prove that there is an infinite number of integrating factors, multiply the last equation by $f(\phi)$, any function of ϕ. Then

$$\mu f(\phi)\,Mdx + \mu f(\phi)\,Ndy = f(\phi)\,d\phi.$$

But $f(\phi)\,d\phi$ is the exact differential of the function $\int f(\phi)\,d\phi$. Hence $\mu f(\phi)$ is an integrating factor of $Mdx + Ndy$.

Ex. 2. *Heat.* It has been noticed (§ 146, Ex. 3) that $\int dQ$ around a closed path is not zero, and consequently dQ is not an exact differential. It is found in the theory of heat that $\int \frac{dQ}{t}$ around a closed path is zero. Hence $\frac{dQ}{t}$ is an exact differential, and we may write $\frac{dQ}{t} = d\phi$. The function ϕ, thus defined, is called the *entropy*.

149. No general method is known for finding integrating factors, but the factors are known for certain cases. We give a list of the simpler cases, leaving it as an exercise for the student to verify by differentiation that each of the equations mentioned satisfies the condition for an exact differential equation after it is multiplied by the proper factor.

1. If $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=f(x)$, then $e^{\int f(x)dx}$ is an integrating factor of $Mdx+Ndy=0$.

2. If $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{M}=f(y)$, then $e^{-\int f(y)dy}$ is an integrating factor of $Mdx+Ndy=0$.

3. If M and N are homogeneous and of the same degree, then $\dfrac{1}{xM+yN}$ is an integrating factor of $Mdx+Ndy=0$.

4. If $M=yf_1(xy)$, $N=xf_2(xy)$; then $\dfrac{1}{xM-yN}$ is an integrating factor of $Mdx+Ndy=0$.

5. $e^{\int f_1(x)\,dx}$ is an integrating factor of the linear equation $\dfrac{dy}{dx}+yf_1(x)=f_2(x)$.

As a practical point, the student should look for an integrating factor only after he has tried to integrate by other methods.

Ex. 1. $(4x^2y-3y^2)\,dx+(x^3-3xy)\,dy=0$.

Here
$$\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=\frac{1}{x}.$$

Consequently $e^{\int\frac{dx}{x}}=x$ is an integrating factor. After multiplication by the factor, the equation becomes
$$(4x^3y-3xy^2)\,dx+(x^4-3x^2y)\,dy=0,$$
the integral of which is
$$x^4y-\tfrac{3}{2}x^2y^2=c.$$

Ex. 2. $(x^2-y^2)\,dx+2xydy=0$.

Since this equation is homogeneous, it has the integrating factor
$$\frac{1}{xM+yN}=\frac{1}{x^3+xy^2}.$$
After multiplication by the factor, the equation becomes
$$\frac{x^2-y^2}{x^3+xy^2}dx+\frac{2xy}{x^3+xy^2}dy=0,$$
or
$$\left(\frac{2x}{x^2+y^2}-\frac{1}{x}\right)dx+\frac{2y}{x^2+y^2}dy=0,$$
the solution of which is $\log(x^2+y^2)-\log x=c'$,

or
$$x+\frac{y^2}{x}=c.$$

For this equation we have also

$$\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=-\frac{2}{x}.$$

Hence it has also the integrating factor

$$e^{\int -\frac{2}{x}dx}=\frac{1}{x^2}.$$

After multiplication by this, the equation becomes

$$\left(1-\frac{y^2}{x^2}\right)dx+\frac{2y}{x}dy=0,$$

the solution of which is $$x+\frac{y^2}{x}=c.$$

The equation may also be solved by the substitution $y = vx$ (§ 78).

150. Stokes's theorem. The theorems of §§ 144–145, which are limited to a plane, may be extended to space as follows: If P, Q, R are three functions of x, y, and z, the line integral

$$\int (Pdx + Qdy + Rdz)$$

along a space curve is defined in a manner precisely similar to the definition of § 143. Let the integral be taken around a closed curve C (fig. 104) and let a surface S be bounded by C. Let dS be the element of area of the surface and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ be the direction cosines of its normal. Then

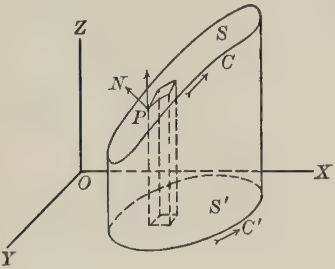

FIG. 104

$$\iint_{(S)}\left[\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)\cos\alpha+\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)\cos\beta+\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)\cos\gamma\right]dS$$
$$=-\int_{(C)}(Pdx+Qdy+Rdz),$$

where the double integral is taken over the surface S and the single integral is taken around C, and the direction of the line integration and that of the normal to S have the relation of fig. 104.

To prove this, let $z = f(x, y)$ be the equation of S, and consider

$$-\int_{(C)} P(x, y, z)\,dx.$$

Then, since $P(x, y, z) = P[x, y, f(x, y)] = P_1(x, y)$, the values of P which correspond to points on C are the same as the values of P_1 which correspond to points on C', the projection of C on the plane XOY. Hence

$$-\int_{(C)} P\,dx = -\int_{(C')} P_1\,dx. \tag{1}$$

But by § 144

$$-\int_{(C')} P_1\,dx = \iint_{(S')} \frac{\partial P_1}{\partial y}\,dxdy, \tag{2}$$

where S' is the projection of S on the plane XOY.

Now

$$\frac{\partial P_1}{\partial y} = \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y},$$

where the right hand of this equation is computed for points on S. Hence from (1) and (2) we have

$$-\int_{(C)} P\,dx = \iint_{(S)} \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y}\right) dxdy. \tag{3}$$

But (§ 112) $\frac{\partial z}{\partial x} : \frac{\partial z}{\partial y} : -1 = \cos\alpha : \cos\beta : \cos\gamma$, and (§ 92) $dxdy = \cos\gamma\, dS$. Substituting in (3), we have

$$-\int_{(C)} P\,dx = \iint_{(S)} \left(\frac{\partial P}{\partial y}\cos\gamma - \frac{\partial P}{\partial z}\cos\beta\right) dS.$$

Similarly,

$$-\int_{(C)} Q\,dy = \iint_{(S)} \left(\frac{\partial Q}{\partial z}\cos\alpha - \frac{\partial Q}{\partial x}\cos\gamma\right) dS,$$

$$-\int_{(C)} R\,dz = \iint_{(S)} \left(\frac{\partial R}{\partial x}\cos\beta - \frac{\partial R}{\partial y}\cos\alpha\right) dS.$$

Strictly speaking, the differential dS is not the same in these three results, since the same element will not project into $dxdy$, $dydz$, and $dzdx$ on the three coördinate planes. But since in a double integral the element of area may be taken at pleasure without changing the value of the integral (§ 129), we may take dS as equal in the three integrals. Adding the equations, we have the required result.

In the above proof we have tacitly assumed that only one point of the surface S is over each point of the coördinate planes, and that α, β, γ are acute angles. The student may show that these restrictions are unessential.

151. From the preceding discussion we derive the following theorem:

If P, Q, and R are three functions of x, y, and z, such that

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}, \qquad \frac{\partial Q}{\partial z}=\frac{\partial R}{\partial y}, \qquad \frac{\partial R}{\partial x}=\frac{\partial P}{\partial z},$$

and the discussion is restricted to a portion of space in which the functions and their derivatives are continuous and one-valued, then

1. *The line integral $\int(Pdx+Qdy+Rdz)$ around any closed curve is zero.*

2. *The line integral $\int(Pdx+Qdy+Rdz)$ between any two points is independent of the path.*

3. *There exists a function $\phi(x, y, z)$, such that*

$$\frac{\partial \phi}{\partial x}=P, \qquad \frac{\partial \phi}{\partial y}=Q, \qquad \frac{\partial \phi}{\partial z}=R,$$

and $Pdx+Qdy+Rdz$ is an exact differential $d\phi$.

Conversely, if any one of the above conclusions is fulfilled, then

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}, \qquad \frac{\partial Q}{\partial z}=\frac{\partial R}{\partial y}, \qquad \frac{\partial R}{\partial x}=\frac{\partial P}{\partial z}.$$

The proof is as in § 145, and is left to the student.

PROBLEMS

1. Find, by the method of Ex. 4, § 143, the area of the four-cusped hypocycloid $x=a\cos^3\phi$, $y=a\sin^3\phi$.

2. Find, by the method of Ex. 4, § 143, the area between one arch of a hypocycloid and the fixed circle.

3. Find, by the method of Ex. 4, § 143, the area between one arch of an epicycloid and the fixed circle.

4. Show that the formula for the area in Ex. 4, § 143, includes, as a special case, the formula for area in polar coördinates.

5. Show that the integral $\int_{(0,0)}^{(1,2)}[2x(x+2y)\,dx + (2x^2 - y^2)\,dy]$ is independent of the path, and find its value.

6. Show that the integral $\int_{(5,0)}^{(3,4)}(xdx + ydy)$ is independent of the path, and find its value.

7. Find the force function for a force in a plane directed toward the origin, and inversely proportional to the square of the distance from the origin.

8. Find the force function for a force in a plane directed toward the origin and inversely proportional to the distance from the origin.

9. Prove that any force directed toward a center and equal to a function of the distance from the center is conservative.

10. Find the value of $\int_{(0,0)}^{(1,1)}[(y-x)\,dy + ydx]$ along the following curves:

(1) the straight line $x = t$, $y = t$;
(2) the parabola $x = t^2$, $y = t$;
(3) the parabola $x = t$, $y = t^2$;
(4) the cubical parabola $x = t$, $y = t^3$.

11. Find the value of $\int_{(0,0)}^{(1,1)}[(x - y^2)\,dx + 2\,xydy]$ along the following paths:

(1) a straight line between the limits;
(2) the axis of x and $x = 1$;
(3) the axis of y and $y = 1$.

12. Find the value of $\int_{(0,0)}^{(1,2)}[y^2dx + (xy + x^2)\,dy]$ along the following paths:

(1) $y = 2x$;
(2) $y = 2x^2$.

13. Find the value of $\int_{(0,2)}^{(-1,-3)}[(1+y^2)\,dx + (1+x^2)\,dy]$ along the following paths:

(1) $y = 5x + 2$;
(2) $y = -5x^2 + 2$;
(3) $y = \dfrac{2-x}{1+2x}$.

Show that the following differential equations are exact, and integrate them:

14. $(2x - y + 1)\,dx + (2y - x - 1)\,dy = 0.$

15. $\dfrac{1+y^2}{x^3}dx - \dfrac{1+x^2}{x^2}ydy = 0.$

16. $(x+y)^2dx + (x^2 + 2xy + 3y^2)\,dy = 0.$

17. $\dfrac{xdx}{\sqrt{x^2+y^2}} + \left(-1 + \dfrac{y}{\sqrt{x^2+y^2}}\right)dy = 0.$

18. $\dfrac{2x - y}{x^2+y^2}dx + \dfrac{2y + x}{x^2+y^2}dy = 0.$

Solve the following differential equations by means of integrating factors:

19. $(x^2 + y^2)\,dx - \dfrac{2x^3}{3y}\,dy = 0.$

20. $(1 + y + x^2y)\,dx + (x + x^3)\,dy = 0.$

21. $ye^{-\frac{x}{y}}\,dx - \left(xe^{-\frac{x}{y}} + y\right)dy = 0.$

22. $(x^3 - y^3)\,dx + xy^2dy = 0.$

23. $(xy^2 - y)\,dx + (x^2y + x)\,dy = 0.$

24. $(5x^3 - 3xy + 2y^2)\,dx + (2xy - x^2)\,dy = 0.$

25. $dy - (y\tan x - \cos x)\,dx = 0.$

26. $\sin(x + y)\,(dx + dy) - y\cos(x + y)\,dy = 0.$

27. $(y + 2xy^2 + x^2y^3)\,dx + (2x^2y + x)\,dy = 0.$

28. $\dfrac{dy}{dx} + \dfrac{y}{1 + x} = x^2.$

CHAPTER XV

INFINITE SERIES

152. Convergence. The expression

$$a_1 + a_2 + a_3 + a_4 + a_5 + \cdots, \tag{1}$$

where the number of the terms is unlimited, is called an *infinite series.*

An infinite series is said to converge, or to be convergent, when the sum of the first n terms approaches a limit as n increases without limit.

Thus, referring to (1), we may place

$$\begin{aligned} s_1 &= a_1, \\ s_2 &= a_1 + a_2, \\ s_3 &= a_1 + a_2 + a_3, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ s_n &= a_1 + a_2 + a_3 + \cdots + a_n. \end{aligned}$$

Then, if $$\lim_{n=\infty} s_n = A,$$

the series is said to converge to the limit A. The quantity A is frequently called the sum of the series, although, strictly speaking, it is the limit of the sum of the first n terms. The convergence of (1) may be seen graphically by plotting $s_1, s_2, s_3, \cdots, s_n$ on the number scale as in I, § 53.

A series which is not convergent is called divergent. This may happen in two ways: either the sum of the first n terms increases without limit as n increases without limit; or s_n may fail to approach a limit, but without becoming indefinitely great.

Ex. 1. Consider the geometric series

$$a + ar + ar^2 + ar^3 + \cdots.$$

Here $s_n = a + ar + ar^2 + \cdots + ar^{n-1} = a\dfrac{1 - r^n}{1 - r}$. Now if r is numerically less than 1, r^n approaches zero as a limit as n increases without limit; and therefore $\lim\limits_{n=\infty} s_n = \dfrac{a}{1-r}$. If, however, r is numerically greater than 1, r^n increases without limit as n increases without limit; and therefore s_n increases without limit. If $r = 1$, the series is

$$a + a + a + a + \cdots,$$

and therefore s_n increases without limit with n. If $n = -1$, the series is

$$a - a + a - a + \cdots,$$

and s_n is alternately a and 0, and hence does not approach a limit.

Therefore, *the geometric series converges to the limit* $\dfrac{a}{1-r}$ *when r is numerically less than unity, and diverges when r is numerically equal to or greater than unity.*

Ex. 2. Consider the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots + \frac{1}{n} + \cdots,$$

consisting of the sum of the reciprocals of the positive integers. Now

$$\tfrac{1}{3} + \tfrac{1}{4} > \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}, \qquad \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} > \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} = \tfrac{1}{2},$$

and in this way the sum of the first n terms of the series may be seen to be greater than any multiple of $\frac{1}{2}$ for a sufficiently large n. Hence *the harmonic series diverges.*

153. Comparison test for convergence. *If each term of a given series of positive numbers is less than, or equal to, the corresponding term of a known convergent series, the given series converges.*

If each term of a given series is greater than, or equal to, the corresponding term of a known divergent series of positive numbers, the given series diverges.

Let

$$a_1 + a_2 + a_3 + a_4 + \cdots \tag{1}$$

be a given series in which each term is a positive number, and let

$$b_1 + b_2 + b_3 + b_4 + \cdots \tag{2}$$

be a known convergent series such that $a_k \leqq b_k$.

Then if s_n is the sum of the first n terms of (1), s'_n the sum of the first n terms of (2), and B the limit of s'_n, it follows that

$$s_n \leqq s'_n < B,$$

since all terms of (1), and therefore of (2), are positive. Now as n increases, s_n increases but always remains less than B. Hence s_n approaches a limit, which is either less than, or equal to, B.

The first part of the theorem is now proved; the second part is too obvious to need formal proof.

In applying this test it is not necessary to begin with the first term of either series, but with any convenient term. The terms before those with which comparison begins, form a polynomial the value of which is of course finite, and the remaining terms form the infinite series the convergence of which is to be determined.

Ex. 1. Consider

$$1 + \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots + \frac{1}{(n-1)^{n-1}} + \cdots.$$

Each term after the third is less than the corresponding term of the convergent geometric series

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots + \frac{1}{2^{n-1}} + \cdots.$$

Therefore the first series converges.

Ex. 2. Consider

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots + \frac{1}{\sqrt{n}} + \cdots.$$

Each term after the first is greater than the corresponding term of the divergent harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n} + \cdots.$$

Therefore the first series diverges.

154. The ratio test for convergence. *If in a series of positive numbers the ratio of the* $(n+1)$*st term to the nth term approaches a limit* L *as* n *increases without limit: then, if* $L < 1$*, the series converges; if* $L > 1$*, the series diverges; if* $L = 1$*, the series may either diverge or converge.*

Let $$a_1 + a_2 + a_3 + \cdots + a_n + a_{n+1} + \cdots \qquad (1)$$

be a series of positive numbers, and let $\lim\limits_{n=\infty} \frac{a_{n+1}}{a_n} = L$. We have three cases to consider.

1. $L < 1$. Take r any number such that $L < r < 1$. Then, since the ratio $\frac{a_{n+1}}{a_n}$ approaches L as a limit, this ratio must become and remain less than r for sufficiently large values of n. Let the ratio be less than r for the mth and all subsequent terms. Then

$$\begin{aligned} &a_{m+1} < a_m r, \\ &a_{m+2} < a_{m+1} r < a_m r^2, \\ &a_{m+3} < a_{m+2} r < a_m r^3, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

Now compare the series

$$a_m + a_{m+1} + a_{m+2} + a_{m+3} + \cdots \tag{2}$$

with the series

$$a_m + a_m r + a_m r^2 + a_m r^3 + \cdots. \tag{3}$$

Each term of (2) except the first is less than the corresponding term of (3), and (3) is a convergent series since it is a geometric series with its ratio less than unity. Hence (2) converges by the comparison test, and therefore (1) converges.

2. $L > 1$. Since $\dfrac{a_{n+1}}{a_n}$ approaches L as a limit as n increases without limit, this ratio eventually becomes and remains greater than unity. Suppose this happens for the mth and all subsequent terms. Then

$$\begin{aligned} &a_{m+1} > a_m, \\ &a_{m+2} > a_{m+1} > a_m, \\ &a_{m+3} > a_{m+2} > a_m, \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{aligned}$$

Each term of the series (2) is greater than the corresponding term of the divergent series

$$a_m + a_m + a_m + a_m + \cdots. \tag{4}$$

Hence (2) and therefore (1) diverges.

3. $L = 1$. Neither of the preceding arguments is valid, and examples show that in this case the series may either converge or diverge.

Ex. 1. Consider

$$1 + \frac{2}{3} + \frac{3}{3^2} + \frac{4}{3^3} + \frac{5}{3^4} + \cdots + \frac{n}{3^{n-1}} + \cdots.$$

The nth term is $\dfrac{n}{3^{n-1}}$ and the $(n+1)$st term is $\dfrac{n+1}{3^n}$. The ratio of the $(n+1)$st term to the nth term is $\dfrac{n+1}{3n}$, and

$$\lim_{n=\infty} \frac{n+1}{3n} = \lim_{n=\infty} \frac{1 + \frac{1}{n}}{3} = \frac{1}{3}.$$

Therefore the given series converges.

Ex. 2. Consider $1+\dfrac{2^2}{\underline{|2}}+\dfrac{3^3}{\underline{|3}}+\dfrac{4^4}{\underline{|4}}+\cdots+\dfrac{n^n}{\underline{|n}}+\cdots.$

The nth term is $\dfrac{n^n}{\underline{|n}}$ and the $(r+1)$st term is $\dfrac{(n+1)^{n+1}}{\underline{|n+1}}$. The ratio of the $(n+1)$st term to the nth term is $\dfrac{(n+1)^{n+1}}{(n+1)\,n^n}=\left(\dfrac{n+1}{n}\right)^n$, and

$$\operatorname*{Lim}_{n=\infty}\left(\frac{n+1}{n}\right)^n=\operatorname*{Lim}_{n=\infty}\left(1+\frac{1}{n}\right)^n=e. \qquad \text{(I, § 156)}$$

Therefore the given series diverges.

155. Absolute convergence. The *absolute value* of a real number is its arithmetical value independent of its algebraic sign. Thus the absolute value of both $+2$ and -2 is 2. The absolute value of a quantity a is often indicated by $|a|$. It is evident that the absolute value of the sum of n quantities is less than, or equal to, the sum of the absolute values of the quantities.

A series converges when the absolute values of its terms form a convergent series, and is said to converge absolutely.

Let
$$a_1+a_2+a_3+a_4+\cdots \tag{1}$$
be a given series, and
$$|a_1|+|a_2|+|a_3|+|a_4|+\cdots \tag{2}$$
the series formed by replacing each term of (1) by its absolute value. We assume that (2) converges, and wish to show the convergence of (1).

Form the auxiliary series

$$(a_1+|a_1|)+(a_2+|a_2|)+(a_3+|a_3|)+(a_4+|a_4|)+\cdots. \tag{3}$$

The terms of (3) are either zero or twice the corresponding terms of (2). For $a_k=-|a_k|$ when a_k is negative, and $a_k=|a_k|$ when a_k is positive.

Now, by hypothesis, (2) converges, and hence the series

$$2|a_1|+2|a_2|+2|a_3|+2|a_4|+\cdots \tag{4}$$

converges. But each term of (3) is either equal to or less than the corresponding term of (4), and hence (3) converges by the comparison test.

Now let s_n be the sum of the first n terms of (1), s'_n the sum of the first n terms of (2), and s''_n the sum of the first n terms of (3). Then

$$s_n = s''_n - s'_n,$$

and, since s''_n and s'_n approach limits, s_n also approaches a limit. Hence the series (1) converges.

We shall consider in this chapter only absolute convergence. Hence the tests of §§ 153, 154 may be applied, since in testing for absolute convergence all terms are considered positive.

156. The power series. A power series is defined by

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots,$$

where a_0, a_1, a_2, a_3, $\cdots$ are numbers not involving x.

We shall prove the following theorem: *If a power series converges for $x = x_1$, it converges absolutely for any value of x such that $|x| < |x_1|$.*

For convenience, let $|x| = X$, $|a_n| = A_n$, $|x_1| = X_1$. By hypothesis the series

$$a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^3 + \cdots + a_nx_1^n + \cdots \tag{1}$$

converges, and we wish to show that

$$A_0 + A_1X + A_2X^2 + A_3X^3 + \cdots + A_nX^n + \cdots \tag{2}$$

converges if $X < X_1$.

Since (1) converges, all its terms are finite. Consequently there must be numbers which are greater than the absolute value of any term of (1). Let M be one such number. Then we have $A_nX_1^n < M$ for all values of n.

Then

$$A_nX^n = A_nX_1^n\left(\frac{X}{X_1}\right)^n < M\left(\frac{X}{X_1}\right)^n.$$

Each term of the series (2) is therefore less than the corresponding term of the series

$$M + M\left(\frac{X}{X_1}\right) + M\left(\frac{X}{X_1}\right)^2 + M\left(\frac{X}{X_1}\right)^3 + \cdots + M\left(\frac{X}{X_1}\right)^n + \cdots. \tag{3}$$

But (3) is a geometric series, which converges when $X < X_1$. Hence, by the comparison test, (2) converges when $X < X_1$.

From the preceding discussion it follows that a power series may behave as to convergence in one of three ways only:

1. It may converge for all finite values of x (Ex. 1).
2. It may converge for no value of x except $x = 0$ (Ex. 2).
3. It may converge for values of x lying between two finite numbers $-R$ and $+R$, and diverge for all other values of x (Ex. 3).

In any case the values of x for which the series converges are together called the *region of convergence.* If represented on a number scale, the region of convergence in the three cases just enumerated is (1) the entire number scale, (2) the zero point only, (3) a portion of the scale having the zero point as its middle point.

Ex. 1. Consider $1 + \frac{x}{1} + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \cdots + \frac{x^{n-1}}{\underline{|n-1}} + \cdots.$

The nth term is $\frac{x^{n-1}}{\underline{|n-1}}$, the $(n+1)$st term is $\frac{x^n}{\underline{|n}}$, and their ratio is $\frac{x}{n}$. $\operatorname*{Lim}_{n=\infty} \frac{x}{n} = 0$ for any finite value of x. Hence the series converges for any value of x and its region of convergence covers the entire number scale.

Ex. 2. Consider

$$1 + x + \underline{|2}\, x^2 + \underline{|3}\, x^3 + \cdots + \underline{|n-1}\, x^{n-1} + \cdots.$$

The nth term is $\underline{|n-1}\, x^{n-1}$, the $(n+1)$st term is $\underline{|n}\, x^n$, and their ratio is nx. This ratio increases without limit for all values of x except $x = 0$. Therefore the series converges for no value of x except $x = 0$.

Ex. 3. Consider

$$1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots.$$

The nth term is nx^{n-1}, the $(n+1)$st term is $(n+1)x^n$, and their ratio is $\frac{n+1}{n}x$. $\operatorname*{Lim}_{n=\infty} \frac{n+1}{n} x = x \operatorname*{Lim}_{n=\infty} \left(1 + \frac{1}{n}\right) = x$. Hence the series converges when $|x| < 1$ and diverges when $|x| > 1$. The region of convergence lies on the number scale between -1 and $+1$.

A power series defines a function of x for values of x within the region of convergence, and we may write

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots, \tag{4}$$

it being understood that the value of $f(x)$ is the limit of the sum of the series on the right of the equation. We shall denote by $s_n(x)$ the polynomial obtained by taking the first n terms of the series in (4). Thus

$$\begin{aligned} s_1(x) &= a_0, \\ s_2(x) &= a_0 + a_1x, \\ s_3(x) &= a_0 + a_1x + a_2x^2, \\ &\cdots\cdots\cdots \\ s_n(x) &= a_0 + a_1x + \cdots + a_{n-1}x^{n-1}. \end{aligned}$$

Graphically, if we plot the curves $y = s_1(x)$, $y = s_2(x)$, $y = s_3(x)$, etc., we shall have a succession of curves which approximate to the curve of the function $y = f(x)$. These curves we shall call the *approximation curves*, calling $y = s_1(x)$ the first approximation curve, $y = s_2(x)$ the second approximation curve, and so on. The graph of $y = f(x)$ we shall call the *limit curve.*

Ex. 4. Let

$$f(x) = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots.$$

The limit curve is the portion of the hyperbola (fig. 105)

$$y = \frac{1}{1-x}$$

between $x = -1$ and $x = 1$. The first approximation curve is the straight line $y = 1$, the second approximation curve is the straight line $y = 1 + x$, the third approximation curve is the parabola $y = 1 + x + x^2$, etc. In fig. 105 the limit curve is drawn heavy and the first four approximation curves are marked (1), (2), (3), (4). It is to be noticed that the curves, except (1), cannot be distinguished from each other for values of x near zero.

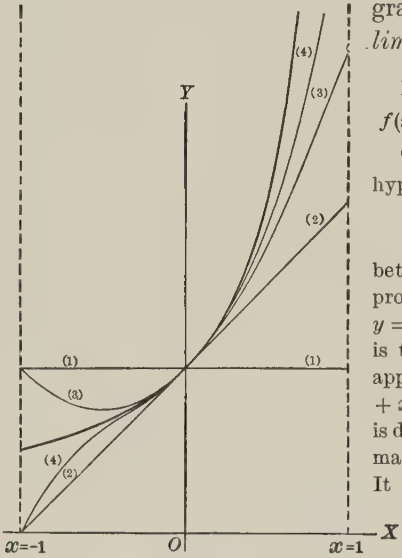

Fig. 105

The power series has the important property, not possessed by all kinds of series, of behaving very similarly to a polynomial. In particular:

1. The function defined by a power series is continuous.
2. The sum, the difference, the product, and the quotient of two functions defined by power series are found by taking the sum, the difference, the product, and the quotient of the series.

3. If $$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots,$$
then
$$\int_a^b f(x)\,dx = \int_a^b a_0 dx + \int_a^b a_1 x dx + \int_a^b a_2x^2 dx + \cdots + \int_a^b a_nx^n dx + \cdots,$$
provided a and b lie in the region of convergence.

4. If $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots,$

then $\quad f'(x) = a_1 + 2\,a_2x + 3\,a_3x^2 + \cdots + na_nx^{n-1} + \cdots,$

and the series for the derivative has the same region of convergence as that for the function.

For proofs of these theorems the student is referred to advanced treatises.*

157. Maclaurin's and Taylor's series. When a function is expressed as a power series, it is possible to express the coefficients of the series in terms of the function and its derivatives. For let
$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots + a_nx^n + \cdots.$$
By differentiating we have
$$\begin{aligned}
f'(x) &= a_1 + 2\,a_2x + 3\,a_3x^2 + 4\,a_4x^3 + \cdots + na_nx^{n-1} + \cdots,\\
f''(x) &= 2\,a_2 + 3\cdot 2\,a_3x + 4\cdot 3\,a_4x^2 + \cdots + n(n-1)\,a_nx^{n-2} + \cdots,\\
f'''(x) &= 3\cdot 2\,a_3 + 4\cdot 3\cdot 2\,a_4x + \cdots + n(n-1)(n-2)\,a_nx^{n-3} + \cdots,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
f^{(n)}(x) &= [n(n-1)(n-2)\cdots 3\cdot 2]\,a_n + \cdots.
\end{aligned}$$

Placing $x = 0$ in each of these equations, we find
$$a_0 = f(0),\ a_1 = f'(0),\ a_2 = \frac{1}{\underline{|2}}f''(0),\ a_3 = \frac{1}{\underline{|3}}f'''(0),\ \cdots,\ a_n = \frac{1}{\underline{|n}}f^{(n)}(0).$$
Consequently we have
$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{\underline{|2}}x^2 + \frac{f'''(0)}{\underline{|3}}x^3 + \cdots + \frac{f^{(n)}(0)}{\underline{|n}}x^n + \cdots. \quad (1)$$
This is Maclaurin's series (§ 31, (6)).

---

* See, for example, Goursat-Hedrick, *Mathematical Analysis*, Chap. IX.

Again, if in the right-hand side of

$$f(x)=a_0+a_1x+a_2x^2+a_3x^3+\cdots+a_nx^n+\cdots,$$

we place $x=a+x'$, and arrange according to powers of x', we have

$$f(x)=b_0+b_1x'+b_2x'^2+b_3x'^3+\cdots+b_nx'^n+\cdots,$$

or, by replacing x' by its value $x-a$,

$$f(x)=b_0+b_1(x-a)+b_2(x-a)^2+b_3(x-a)^3+\cdots+b_n(x-a)^n+\cdots.$$

By differentiating this equation successively, and placing $x=a$ in the results, we readily find

$$b_0=f(a),\ b_1=f'(a),\ b_2=\frac{1}{\underline{|2}}f''(a),\ b^3=\frac{1}{\underline{|3}}f'''(a),\ \cdots,\ b^n=\frac{1}{\underline{|n}}f^{(n)}(a).$$

Hence

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{\underline{|2}}(x-a)^2+\frac{f'''(a)}{\underline{|3}}(x-a)^3+\cdots$$
$$+\frac{f^{(n)}(a)}{\underline{|n}}(x-a)^n+\cdots. \qquad (2)$$

This is Taylor's series (§ 31, (4)).

We have here shown that if a function can be expressed as a power series, the series may be put in the form (1) or (2). In § 31 we showed that any function which is continuous and has continuous derivatives can be so expressed. Usually when a known function is expressed by either (1) or (2), the region in which the expression is valid as a representation of the function is coincident with the region of convergence of the series. Examples can be given in which this is not true, but the student is not likely to meet them in practice.

158. Taylor's series for functions of several variables. Consider $f(x, y)$ a function of two variables x and y. If we place

$$x=a+lt, \qquad y=b+mt,$$

where a, b, l, and m are constants and t is a variable, we have

$$f(x, y)=f(a+lt, b+mt)=F(t).$$

Now, by expansion into Maclaurin's series,

$$F(t)=F(0)+F'(0)t+\frac{F''(0)}{\underline{|2}}t^2+\frac{F'''(0)}{\underline{|3}}t^3+\cdots, \tag{1}$$

and, by §§ 111 and 118,

$$F'(t)=\frac{\partial f}{\partial x}l+\frac{\partial f}{\partial y}m,$$

$$F''(t)=\frac{\partial^2 f}{\partial x^2}l^2+2\frac{\partial^2 f}{\partial x\partial y}lm+\frac{\partial^2 f}{\partial y^2}m^2,$$

$$F'''(t)=\frac{\partial^3 f}{\partial x^3}l^3+3\frac{\partial^3 f}{\partial x^2\partial y}l^2m+3\frac{\partial^3 f}{\partial x\partial y^2}lm^2+\frac{\partial^3 f}{\partial y^3}m^3.$$

When $t=0$, we have $x=a$ and $y=b$. Hence $F(0)=f(a, b)$, and, if we denote by a subscript zero the values of the derivatives of $f(x, y)$ when $x=a$, $y=b$,

$$F'(0)=\left(\frac{\partial f}{\partial x}\right)_0 l+\left(\frac{\partial f}{\partial y}\right)_0 m,$$

$$F''(0)=\left(\frac{\partial^2 f}{\partial x^2}\right)_0 l^2+2\left(\frac{\partial^2 f}{\partial x\partial y}\right)_0 lm+\left(\frac{\partial^2 f}{\partial y^2}\right)_0 m^2,$$

$$F'''(0)=\left(\frac{\partial^3 f}{\partial x^3}\right)_0 l^3+3\left(\frac{\partial^3 f}{\partial x^2\partial y}\right)_0 l^2m+3\left(\frac{\partial^3 f}{\partial x\partial y^2}\right)_0 lm^2+\left(\frac{\partial^3 f}{\partial y^3}\right)_0 m^3.$$

By substitution in (1), noting that $lt=x-a$ and $mt=y-b$, we have

$$\begin{aligned}f(x, y)&=f(a, b)+\left(\frac{\partial f}{\partial x}\right)_0(x-a)+\left(\frac{\partial f}{\partial y}\right)_0(y-b)\\&+\frac{1}{\underline{|2}}\left[\left(\frac{\partial^2 f}{\partial x^2}\right)_0(x-a)^2+2\left(\frac{\partial^2 f}{\partial x\partial y}\right)_0(x-a)(y-b)+\left(\frac{\partial^2 f}{\partial y^2}\right)_0(y-b)^2\right]\\&+\frac{1}{\underline{|3}}\left[\left(\frac{\partial^3 f}{\partial x^3}\right)_0(x-a)^3+3\left(\frac{\partial^3 f}{\partial x^2\partial y}\right)_0(x-a)^2(y-b)\right.\\&\left.\quad+3\left(\frac{\partial^3 f}{\partial x\partial y^2}\right)_0(x-a)(y-b)^2+\left(\frac{\partial^3 f}{\partial y^3}\right)_0(y-b)^3\right]+\cdots.\end{aligned} \tag{2}$$

Another form of this series may be obtained by placing $x-a=h$ and $y-b=k$. Then

$$
\begin{aligned}
f(a+h,\, b+k) = f(a,\, b) &+ \left(\frac{\partial f}{\partial x}\right)_0 h + \left(\frac{\partial f}{\partial y}\right)_0 k \\
&+ \frac{1}{\underline{|2}}\left[\left(\frac{\partial^2 f}{\partial x^2}\right)_0 h^2 + 2\left(\frac{\partial^2 f}{\partial x \partial y}\right)_0 hk + \left(\frac{\partial^2 f}{\partial y^2}\right)_0 k^2\right] \\
&+ \frac{1}{\underline{|3}}\left[\left(\frac{\partial^3 f}{\partial x^3}\right)_0 h^3 + 3\left(\frac{\partial^3 f}{\partial x^2 \partial y}\right)_0 h^2 k + 3\left(\frac{\partial^3 f}{\partial x \partial y^2}\right)_0 hk^2 + \left(\frac{\partial^3 f}{\partial y^3}\right)_0 k^3\right] + \cdots. \qquad (3)
\end{aligned}
$$

In a similar manner, we may show that

$$
\begin{aligned}
f(a+h,\, b+k,\, c+l) = f(a,\, b,\, c) &+ \left(\frac{\partial f}{\partial x}\right)_0 h + \left(\frac{\partial f}{\partial y}\right)_0 k + \left(\frac{\partial f}{\partial z}\right)_0 l \\
&+ \frac{1}{\underline{|2}}\left[\left(\frac{\partial^2 f}{\partial x^2}\right)_0 h^2 + \left(\frac{\partial^2 f}{\partial y^2}\right)_0 k^2 + \left(\frac{\partial^2 f}{\partial z^2}\right)_0 l^2 + 2\left(\frac{\partial^2 f}{\partial x \partial y}\right)_0 hk \right. \\
&\left. + 2\left(\frac{\partial^2 f}{\partial y \partial z}\right)_0 kl + 2\left(\frac{\partial^2 f}{\partial z \partial x}\right)_0 lh\right] + \cdots. \qquad (4)
\end{aligned}
$$

The terms of the nth degree in this expansion may be indicated symbolically as

$$\frac{1}{\underline{|n}}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right)^n f_0.$$

159. Fourier's series. A series of the form

$$
\begin{aligned}
&\frac{a_0}{2} + a_1 \cos x + a_2 \cos 2\,x + \cdots + a_n \cos nx + \cdots \\
&\quad + b_1 \sin x + b_2 \sin 2\,x + \ldots + b_n \sin nx + \cdots, \qquad (1)
\end{aligned}
$$

where the coefficients $a_0, a_1, \cdots, b_1, b_2, \cdots$ do not involve x and are determined by the formulas derived in § 160, is called a *Fourier's series.* Every term of (1) has the period* $2\,\pi$, and hence (1) has that period. Accordingly any function defined for all values of x by a Fourier's series must have the period $2\,\pi$. But even if a function does not have the period $2\,\pi$, it is possible to find a Fourier's series which will represent the function for all values of x between $-\pi$ and π, provided that in the interval $-\pi$ to π

---

* $f(x)$ is called a periodic function, with period k, if $f(x+k)=f(x)$.

the function is single-valued, finite, and continuous except for finite discontinuities,* and provided there is not an infinite number of maxima or minima in the neighborhood of any point.

160. We will now try to determine the formulas for the coefficients of a Fourier's series, which, for all values of x between $-\pi$ and π, shall represent a given function, $f(x)$, which satisfies the above conditions.

$$\begin{aligned}\text{Let } f(x) = \frac{a_0}{2} &+ a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx + \cdots \\ &+ b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx + \cdots. \qquad (1)\end{aligned}$$

To determine a_0, multiply (1) by dx and integrate from $-\pi$ to π, term by term. The result is

$$\int_{-\pi}^{\pi} f(x)\,dx = a_0\pi,$$

whence
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \qquad (2)$$

since all the terms on the right-hand side of the equation, except the one involving a_0, vanish.

To determine the coefficient of the general cosine term, as a_n, multiply (1) by $\cos nx\,dx$, and integrate from $-\pi$ to π, term by term. Since for all integral values of m and n

$$\int_{-\pi}^{\pi} \sin mx \cos nx\,dx = 0,$$

$$\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = 0 \qquad (m \neq n)$$

and
$$\int_{-\pi}^{\pi} \cos^2 nx\,dx = \pi,$$

all the terms on the right-hand side of the equation, except the one involving a_n, vanish and the result is

$$\int_{-\pi}^{\pi} f(x) \cos nx\,dx = a_n\pi,$$

whence
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\,dx. \qquad (3)$$

---

* If x_1 is any value of x, such that $f(x_1 - \epsilon)$ and $f(x_1 + \epsilon)$ have different limits as ϵ approaches the limit zero, then $f(x)$ is said to have a finite discontinuity for the value $x = x_1$. Graphically, the curve $y = f(x)$ approaches two distinct points on the ordinate $x = x_1$, one point being approached as x increases toward x_1, and the other being approached as x decreases toward x_1.

It is to be noted that (3) reduces to (2) when $n = 0$.

In like manner, to determine b_n, multiply (1) by $\sin nx\,dx$, and integrate from $-\pi$ to π, term by term. The result is

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\,dx. \tag{4}$$

For a proof of the validity of the above method of deriving the formulas (2), (3), and (4), the reader is referred to advanced treatises.*

Ex. 1. Expand x in a Fourier's series, the development to hold for all values of x between $-\pi$ and π.

By (2),
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x dx = 0,$$

by (3),
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos nx\,dx = 0,$$

and by (4),
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\,dx = -\frac{2}{n}\cos n\pi.$$

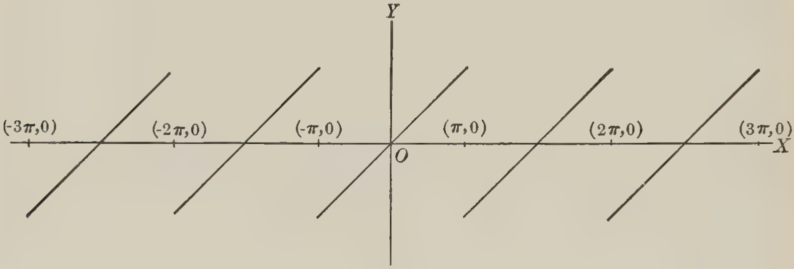

Fig. 106

Hence only the sine terms appear in the series for x, the values of the coefficients being determined by giving n in the expression for b_n the values 1, 2, 3, $\cdots$ in succession. Therefore $b_1 = 2$, $b_2 = -\frac{2}{2}$, $b_3 = \frac{2}{3}, \cdots$, and

$$x = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right).$$

The graph of the function x is the infinite straight line passing through the origin and bisecting the angles of the first and the third quadrants.

The limit curve of the series coincides with this line for all values of x between $-\pi$ and π, but not for $x = -\pi$ and $x = \pi$; for every term of the series vanishes when $x = -\pi$ or $x = \pi$, and therefore the graph of the series has the points $(\pm\pi, 0)$ as isolated points (fig. 106).

By taking x_1 as any value of x between $-\pi$ and π, and giving k the values 1, 2, 3, $\cdots$ in succession, we can represent all values of x by $x_1 \pm 2k\pi$. But the series has the period 2π, and accordingly has the same value for $x_1 \pm 2k\pi$ as for x_1. Hence the limit curve is a series of repetitions of the part between $x = -\pi$ and $x = \pi$, and the isolated points $(\pm 2k\pi, 0)$.

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* See, for example, Goursat-Hedrick, *Mathematical Analysis*, Chap. IX.

It should be noted that the function defined by the series has finite discontinuities, while the function from which the series is derived is continuous.

It is not necessary that $f(x)$ should be defined by the same law throughout the interval from $-\pi$ to π. In this case the integrals defining the coefficients break up into two or more integrals, as shown in the following examples:

Ex. 2. Find the Fourier's series for $f(x)$ for all values of x between $-\pi$ and π, where $f(x) = x + \pi$ if $-\pi < x < 0$, and $f(x) = \pi - x$ if $0 < x < \pi$.

Here
$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0}(x+\pi)\,dx + \int_{0}^{\pi}(\pi - x)\,dx\right] = \pi;$$
$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(x+\pi)\cos nx\,dx + \int_{0}^{\pi}(\pi - x)\cos nx\,dx\right]$$
$$= \frac{2}{\pi n^2}(1 - \cos n\pi);$$
$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(x+\pi)\sin nx\,dx + \int_{0}^{\pi}(\pi - x)\sin nx\,dx\right]$$
$$= 0.$$

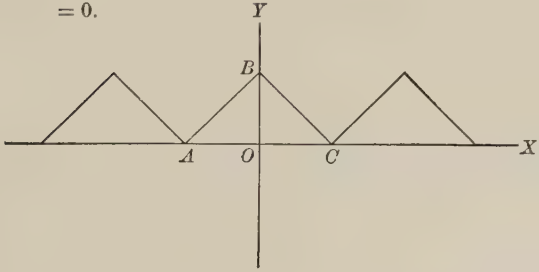

Fig. 107

Therefore the required series is

$$\frac{\pi}{2} + \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right).$$

The graph of $f(x)$ for values of x between $-\pi$ and π is the broken line ABC (fig. 107). When $x = 0$, the series reduces to $\frac{\pi}{2} + \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) = \pi$, for $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.* When $x = \pm\pi$, the series reduces to 0. Hence the limit curve of the series coincides with the broken line ABC at all points. From the periodicity of the series it is seen, as in Ex. 1, that the limit curve is the broken line of fig. 107.

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* Byerly, *Fourier's Series*, p. 40.

Ex. 3. Find the Fourier's series for $f(x)$, for all values of x between $-\pi$ and π, where $f(x)=0$ if $-\pi<x<0$, and $f(x)=\pi$ if $0<x<\pi$.

Here
$$a_0=\frac{1}{\pi}\left(\int_{-\pi}^{0}0\,dx+\int_0^{\pi}\pi\,dx\right)=\pi;$$
$$a_n=\frac{1}{\pi}\int_0^{\pi}\pi\cos nx\,dx=0;$$
$$b_n=\frac{1}{\pi}\int_0^{\pi}\pi\sin nx\,dx=\frac{1}{n}(1-\cos n\pi).$$

Therefore the required series is
$$\frac{\pi}{2}+2\left(\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\cdots\right).$$

The graph of the function for the values of x between $-\pi$ and π is the axis of x from $x=-\pi$ to $x=0$, and the straight line AB (fig. 108), there being a finite discontinuity when $x=0$.

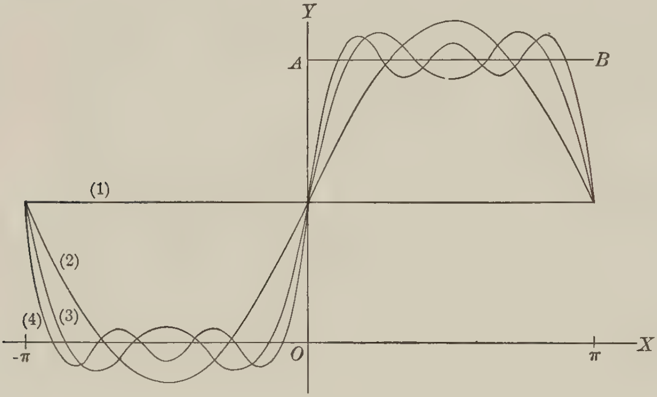

FIG. 108

The curves (1), (2), (3), and (4) are the approximation curves corresponding respectively to the equations

$$y=\frac{\pi}{2}, \tag{1}$$
$$y=\frac{\pi}{2}+2\sin x, \tag{2}$$
$$y=\frac{\pi}{2}+2\left(\frac{\sin x}{1}+\frac{\sin 3x}{3}\right), \tag{3}$$
$$y=\frac{\pi}{2}+2\left(\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}\right). \tag{4}$$

They may be readily constructed by the method used in I, § 149, Ex. 5. It is to be noted that all the curves pass through the point $\left(0,\frac{\pi}{2}\right)$, which is midway between the points A and O, which correspond to the finite discontinuity, and that the successive curves approach perpendicularity to the axis of x at that point.

161. The indeterminate form $\frac{0}{0}$. Consider the fraction

$$\frac{f(x)}{\phi(x)}, \tag{1}$$

and let a be a number such that $f(a)=0$ and $\phi(a)=0$. If we place $x=a$ in (1), we obtain the expression $\frac{0}{0}$, which is literally meaningless.

It is customary, however, to *define* the *value* of the fraction (1) when $x=a$, as the *limit* approached by the fraction as x approaches a.

In some cases this limit can be found by elementary methods.

Ex. 1. $\dfrac{a^2-x^2}{a-x}$.

When $x=a$, this becomes $\frac{0}{0}$. When $x \neq a$, we may divide both terms of the fraction by $a-x$, and have

$$\frac{a^2-x^2}{a-x} = a+x \tag{1}$$

for all values of x except $x=a$. Equation (1) is true as x approaches a, and hence

$$\operatorname*{Lim}_{x \doteq a} \frac{a^2-x^2}{a-x} = \operatorname*{Lim}_{x \doteq a}(a+x) = 2a.$$

Ex. 2. $\dfrac{1-\sqrt{1-x^2}}{x}$.

When $x=0$, this becomes $\frac{0}{0}$. When $x \neq 0$, we have

$$\frac{1-\sqrt{1-x^2}}{x} = \frac{1-\sqrt{1-x^2}}{x} \cdot \frac{1+\sqrt{1-x^2}}{1+\sqrt{1-x^2}} = \frac{x}{1+\sqrt{1-x^2}}.$$

Hence $$\operatorname*{Lim}_{x \doteq 0} \frac{1-\sqrt{1-x^2}}{x} = \operatorname*{Lim}_{x \doteq 0} \frac{x}{1+\sqrt{1-x^2}} = 0.$$

Ex. 3. $\dfrac{e^x-1}{\sin x}$.

When $x=0$, this becomes $\frac{0}{0}$. When $x \neq 0$, we may expand each term by Maclaurin's series, and have

$$\frac{e^x-1}{\sin x} = \frac{x + \dfrac{x^2}{\underline{|2}} + \dfrac{x^3}{\underline{|3}} + \cdots}{x - \dfrac{x^3}{\underline{|3}} + \dfrac{x^5}{\underline{|5}} - \cdots}$$

$$= \frac{1 + \dfrac{x}{\underline{|2}} + \dfrac{x^2}{\underline{|3}} + \cdots}{1 - \dfrac{x^2}{\underline{|3}} + \dfrac{x^4}{\underline{|5}} - \cdots}.$$

Hence $$\operatorname*{Lim}_{x \doteq 0} \frac{e^x-1}{\sin x} = 1.$$

The method by expansion into series, used in Ex. 3, suggests a general method. Since we wish to determine the limit approached by $\frac{f(x)}{\phi(x)}$ as x approaches a, we will place $x = a + h$ and expand by Taylor's series. We have (§ 31, (5))

$$\frac{f(x)}{\phi(x)} = \frac{f(a+h)}{\phi(a+h)} = \frac{f(a) + hf'(a) + \frac{h^2}{\underline{|2}} f''(a) + \frac{h^3}{\underline{|3}} f'''(a) + \cdots}{\phi(a) + h\phi'(a) + \frac{h^2}{\underline{|2}} \phi''(a) + \frac{h^3}{\underline{|3}} \phi'''(a) + \cdots}.$$

But by hypothesis $f(a) = 0$ and $\phi(a) = 0$. Therefore, since $h \neq 0$,

$$\frac{f(x)}{\phi(x)} = \frac{f'(a) + \frac{h}{\underline{|2}} f''(a) + \frac{h^2}{\underline{|3}} f'''(a) + \cdots}{\phi'(a) + \frac{h}{\underline{|2}} \phi''(a) + \frac{h^2}{\underline{|3}} \phi'''(a) + \cdots}. \tag{2}$$

Now as $x \doteq a$, $h \doteq 0$, and hence, unless $f'(a) = 0$ and $\phi'(a) = 0$, we have from (2)

$$\operatorname*{Lim}_{x \doteq a} \frac{f(x)}{\phi(x)} = \frac{f'(a)}{\phi'(a)}. \tag{3}$$

If, however, $f'(a) = 0$ and $\phi'(a) = 0$, the right-hand side of (3) becomes $\frac{0}{0}$. In this case (2) becomes

$$\frac{f(x)}{\phi(x)} = \frac{f''(a) + \frac{h}{3} f'''(a) + \cdots}{\phi''(a) + \frac{h}{3} \phi'''(a) + \cdots},$$

whence

$$\operatorname*{Lim}_{x \doteq a} \frac{f(x)}{\phi(x)} = \frac{f''(a)}{\phi''(a)},$$

unless $f''(a) = 0$ and $\phi''(a) = 0$. In the latter case we may go back again to (2) and repeat the reasoning.

Accordingly we have the rule:

To find the value of a fraction which takes the form $\frac{0}{0}$ *when* $x = a$, *replace the numerator and the denominator each by its derivative and substitute* $x = a$. *If the new fraction is also* $\frac{0}{0}$, *repeat the process.*

Ex. 4. To find the limit approached by $\frac{e^x - e^{-x}}{\sin x}$ when $x \doteq 0$.

By the rule,
$$\operatorname*{Lim}_{x \doteq 0} \frac{e^x - e^{-x}}{\sin x} = \left[\frac{e^x + e^{-x}}{\cos x}\right]_{x=0} = \frac{2}{1} = 2.$$

Ex. 5. To find the limit approached by $\frac{e^x - 2\cos x + e^{-x}}{x \sin x}$ when $x \doteq 0$.

If we apply the rule once, we have

$$\operatorname*{Lim}_{x \doteq 0} \frac{e^x - 2\cos x + e^{-x}}{x \sin x} = \left[\frac{e^x + 2\sin x - e^{-x}}{\sin x + x\cos x}\right]_{x=0} = \frac{0}{0}.$$

We therefore apply the rule again, thus:

$$\operatorname*{Lim}_{x \doteq 0} \frac{e^x - 2\cos x + e^{-x}}{x \sin x} = \left[\frac{e^x + 2\cos x + e^{-x}}{2\cos x - x\sin x}\right]_{x=0} = \frac{4}{2} = 2.$$

162. The derivation of the rule in § 161 assumes the possibility of the expansion of $f(x)$ and $\phi(x)$ into power series. It is, however, sometimes necessary to consider functions for which this assumption is not valid. We shall accordingly give the following new proof of the rule.

Consider formula (3), § 30, namely,

$$F(b) - F(a) = (b - a)F'(\xi).$$

From this it follows that if $F(b) = 0$ and $F(a) = 0$, then $F'(\xi) = 0$, where ξ is some number between a and b. Let us apply this to the arbitrarily formed function

$$F(x) = \frac{f(b) - f(a)}{\phi(b) - \phi(a)}[\phi(x) - \phi(a)] - [f(x) - f(a)],$$

where $F(b) = 0$ and $F(a) = 0$, as may be seen by direct substitution. Then

$$0 = F'(\xi) = \frac{f(b) - f(a)}{\phi(b) - \phi(a)}\phi'(\xi) - f'(\xi),$$

whence
$$\frac{f(b) - f(a)}{\phi(b) - \phi(a)} = \frac{f'(\xi)}{\phi'(\xi)}. \qquad (1)$$

Now, in (1), let $f(a) = 0$, $\phi(a) = 0$, and place $b = a + h$. We have

$$\frac{f(a + h)}{\phi(a + h)} = \frac{f'(\xi)}{\phi'(\xi)}. \qquad (2)$$

As $h \doteq 0$, $\xi \doteq a$, since $a < \xi < a + h$. Hence

$$\operatorname*{Lim}_{h \doteq 0} \frac{f(a+h)}{\phi(a+h)} = \operatorname*{Lim}_{\xi \doteq a} \frac{f'(\xi)}{\phi'(\xi)},$$

which may be written $$\operatorname*{Lim}_{x \doteq a} \frac{f(x)}{\phi(x)} = \operatorname*{Lim}_{x \doteq a} \frac{f'(x)}{\phi'(x)}. \tag{3}$$

If $f'(a)$ and $\phi'(a)$ are not both zero, (3) gives

$$\operatorname*{Lim}_{x \doteq a} \frac{f(x)}{\phi(x)} = \frac{f'(a)}{\phi'(a)}.$$

If, however, $f'(a) = 0$ and $\phi'(a) = 0$, (3) gives

$$\operatorname*{Lim}_{x \doteq a} \frac{f(x)}{\phi(x)} = \operatorname*{Lim}_{x \doteq a} \frac{f'(x)}{\phi'(x)} = \operatorname*{Lim}_{x \doteq a} \frac{f''(x)}{\phi''(x)}.$$

The rule of § 161 thus results.

163. The indeterminate form $\frac{\infty}{\infty}$. If $f(a) = \infty$ and $\phi(a) = \infty$, the fraction $\frac{f(x)}{\phi(x)}$ takes the meaningless form $\frac{\infty}{\infty}$. The value of the fraction is then *defined* as the limit approached by the fraction as x approaches a as a limit. We shall now prove that *the rule for finding the value of a fraction which becomes* $\frac{0}{0}$ *holds also for a fraction which becomes* $\frac{\infty}{\infty}$.

To prove this, we shall take first the case in which the fraction $\frac{f(x)}{\phi(x)}$ becomes $\frac{\infty}{\infty}$ when $x = \infty$. By placing $b = x$, $a = c$ in (1), § 162, we have

$$\frac{f(x) - f(c)}{\phi(x) - \phi(c)} = \frac{f'(\xi)}{\phi'(\xi)}, \qquad (c < \xi < x)$$

which is equivalent to

$$\frac{f(x)}{\phi(x)} = \frac{f'(\xi)}{\phi'(\xi)} \cdot \frac{1 - \dfrac{\phi(c)}{\phi(x)}}{1 - \dfrac{f(c)}{f(x)}}. \tag{1}$$

Now we assume that $\frac{f'(x)}{\phi'(x)}$ approaches a limit k as $x = \infty$. Hence we may take c so great that $\frac{f'(c)}{\phi'(c)}$, and therefore $\frac{f'(\xi)}{\phi'(\xi)}$, differs from

k by as little as we please. This fixes c; we may then take x so much greater than c that $\frac{\phi(c)}{\phi(x)}$ and $\frac{f(c)}{f(x)}$ may each differ from zero by as little as we please. By proper choice of c and x, then, we have, from (1),

$$\frac{f(x)}{\phi(x)}=(k+\epsilon_1)(1+\epsilon_2),$$

where ϵ_1 and ϵ_2 may be as small as we please. Hence

$$\operatorname*{Lim}_{x=\infty}\frac{f(x)}{\phi(x)}=k=\operatorname*{Lim}_{x=\infty}\frac{f'(x)}{\phi'(x)}. \tag{2}$$

To extend this result to the case in which $\frac{f(x)}{\phi(x)}$ becomes $\frac{\infty}{\infty}$ when $x=a$, we place $x=a+\frac{1}{y}\cdot$ Then

$$\frac{f(x)}{\phi(x)}=\frac{f\left(a+\frac{1}{y}\right)}{\phi\left(a+\frac{1}{y}\right)}=\frac{F(y)}{\Phi(y)}.$$

Then $\frac{F(y)}{\Phi(y)}$ becomes $\frac{\infty}{\infty}$ when $y=\infty$. Therefore, from (2),

$$\operatorname*{Lim}_{y=\infty}\frac{F(y)}{\Phi(y)}=\operatorname*{Lim}_{y=\infty}\frac{F'(y)}{\Phi'(y)}.$$

Now $$F'(y)=f'(x)\frac{dx}{dy}=-\frac{1}{y^2}f'(x),$$

and $$\Phi'(y)=\phi'(x)\frac{dx}{dy}=-\frac{1}{y^2}\phi'(x).$$

Hence $\frac{F'(y)}{\Phi'(y)}=\frac{f'(x)}{\phi'(x)}$, and $\operatorname*{Lim}_{y=\infty}\frac{F'(y)}{\Phi'(y)}=\operatorname*{Lim}_{x\doteq a}\frac{f'(x)}{\phi'(x)}\cdot$ Hence we have

$$\operatorname*{Lim}_{x\doteq a}\frac{f(x)}{\phi(x)}=\operatorname*{Lim}_{x\doteq a}\frac{f'(x)}{\phi'(x)}. \tag{3}$$

From (2) and (3) follows the proposition that we wished to prove.

Ex. To find the limit approached by $\frac{\log x}{x^n}$ as x becomes infinite.

By the rule, $$\operatorname*{Lim}_{x=\infty}\frac{\log x}{x^n}=\operatorname*{Lim}_{x=\infty}\frac{\frac{1}{x}}{nx^{n-1}}=\operatorname*{Lim}_{x=\infty}\frac{1}{nx^n}=0.$$

164. Other indeterminate forms. There are other indeterminate forms indicated by the symbols

$$0 \cdot \infty, \qquad \infty - \infty, \qquad 0^0, \qquad \infty^0, \qquad 1^\infty.$$

The form $0 \cdot \infty$ arises when, in a product $f(x) \cdot \phi(x)$, we have $f(a) = 0$ and $\phi(a) = \infty$. The form $\infty - \infty$ arises when, in $f(x) - \phi(x)$, we have $f(a) = \infty$, $\phi(a) = \infty$.

These forms are handled by expressing $f(x) \cdot \phi(x)$ or $f(x) - \phi(x)$, as the case may be, in the form of a fraction which becomes $\frac{0}{0}$ or $\frac{\infty}{\infty}$ when $x = a$. The rule of § 161 may then be applied.

Ex. 1. $x^3 e^{-x^2}$.

When $x = \infty$, this becomes $\infty \cdot 0$. We have, however, $x^3 e^{-x^2} = \frac{x^3}{e^{x^2}}$, which becomes $\frac{\infty}{\infty}$ when $x = \infty$. Then

$$\operatorname*{Lim}_{x=\infty} \frac{x^3}{e^{x^2}} = \operatorname*{Lim}_{x=\infty} \frac{3x^2}{2xe^{x^2}} = \operatorname*{Lim}_{x=\infty} \frac{3x}{2e^{x^2}} = \operatorname*{Lim}_{x=\infty} \frac{3}{4xe^{x^2}} = 0.$$

In the same manner $\operatorname*{Lim}_{x=\infty} x^n e^{-x^2} = 0$

for any value of n.

Ex. 2. $\sec x - \tan x$.

When $x = \frac{\pi}{2}$, this is $\infty - \infty$. We have, however,

$$\sec x - \tan x = \frac{1 - \sin x}{\cos x},$$

which becomes $\frac{0}{0}$ when $x = \frac{\pi}{2}$. Then

$$\operatorname*{Lim}_{x \doteq \frac{\pi}{2}} (\sec x - \tan x) = \operatorname*{Lim}_{x \doteq \frac{\pi}{2}} \frac{1 - \sin x}{\cos x} = \operatorname*{Lim}_{x \doteq \frac{\pi}{2}} \frac{-\cos x}{-\sin x} = 0.$$

The forms 0^0, ∞^0, 1^∞ may arise for the function

$$[f(x)]^{\phi(x)}$$

when $x = a$.

If we place $u = [f(x)]^{\phi(x)}$,

we have $\log u = \phi(x) \log f(x)$.

If $\operatorname*{Lim}_{x \doteq a} \phi(x) \log f(x)$ can be obtained by the previous methods, the limit approached by u can be found.

Ex. 3. $(1-x)^{\frac{1}{x}}$.

When $x=0$, this becomes 1^{∞}. Place

$$u=(1-x)^{\frac{1}{x}};$$

then

$$\log u=\frac{\log(1-x)}{x}.$$

Now, by § 161, $\quad \operatorname*{Lim}_{x \doteq 0} \dfrac{\log(1-x)}{x}=\left[\dfrac{-1}{1-x}\right]_{x=0}=-1.$

Hence $\log u$ approaches the limit -1 and u approaches the limit $\frac{1}{e}$.

PROBLEMS

1. Prove that the series

$$1+\frac{1}{2^a}+\frac{1}{2^a}+\frac{1}{4^a}+\frac{1}{4^a}+\frac{1}{4^a}+\frac{1}{4^a}+\frac{1}{8^a}+\cdots,$$

where there are two terms of the form $\frac{1}{2^a}$, four terms of the form $\frac{1}{4^a}$, eight terms of the form $\frac{1}{8^a}$, and 2^k terms of the form $\frac{1}{(2^k)^a}$, $(k=1, 2, 3, \cdots)$, converges when $a>1$.

2. By comparison with the series in Ex. 1 or with the harmonic series (Ex. 2, § 152), prove that the series

$$1+\frac{1}{2^a}+\frac{1}{3^a}+\frac{1}{4^a}+\cdots+\frac{1}{n^a}+\cdots$$

converges when $a>1$, and diverges when $a \leqq 1$.

Test the following series for convergence or divergence:

3. $2+\dfrac{2^3}{3}+\dfrac{2^5}{5}+\cdots+\dfrac{2^{2n-1}}{2n-1}+\cdots.$

4. $1+\dfrac{1}{3}+\dfrac{1}{5\underline{|2}}+\dfrac{1}{7\underline{|3}}+\cdots+\dfrac{1}{(2n-1)\underline{|n-1}}+\cdots.$

5. $\dfrac{1}{2^2}+\dfrac{1}{4^2}+\dfrac{1}{6^2}+\cdots+\dfrac{1}{(2n)^2}+\cdots.$

6. $\dfrac{1}{1\cdot 2}+\dfrac{1}{3\cdot 4}+\dfrac{1}{5\cdot 6}+\cdots+\dfrac{1}{(2n-1)2n}+\cdots.$

7. $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\cdots+\dfrac{1}{n(n+1)}+\cdots.$

8. $\dfrac{3}{1\cdot 2}+\dfrac{3^2}{2\cdot 3}+\dfrac{3^3}{3\cdot 4}+\cdots+\dfrac{3^n}{n(n+1)}+\cdots.$

9. $\frac{1}{4}+\frac{2}{4^2}+\frac{3}{4^3}+\cdots+\frac{n}{4^n}+\cdots.$ **10.** $1+\frac{1}{2^2}+\frac{1}{3^3}+\cdots+\frac{1}{n^n}+\cdots.$

11. $\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}+\cdots+\frac{1}{n(n+1)(n+2)}+\cdots.$

12. $\frac{1}{2}+\frac{1}{5}+\frac{1}{10}+\cdots+\frac{1}{n^2+1}+\cdots.$ **13.** $1+\frac{2^3}{\underline{|2}}+\frac{3^3}{\underline{|3}}+\cdots+\frac{n^3}{\underline{|n}}+\cdots.$

Find the region of convergence of each of the following series:

14. $x+\frac{3}{5}x^2+\frac{4}{10}x^3+\cdots+\frac{n+1}{n^2+1}x^n+\cdots.$

15. $1+2^2x+3^2x^2+\cdots+n^2x^{n-1}+\cdots.$

16. $\sin^{-1}x=x+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5}+\cdots+\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2\cdot 4\cdot 6\cdots(2n-2)}\cdot\frac{x^{2n-1}}{2n-1}+\cdots.$

17. $\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+(-1)^{n+1}\frac{x^n}{n}+\cdots.$

18. $\sin x=x-\frac{x^3}{\underline{|3}}+\frac{x^5}{\underline{|5}}-\cdots+(-1)^{n+1}\frac{x^{2n-1}}{\underline{|2n-1}}+\cdots.$

19. $e^x=1+x+\frac{x^2}{\underline{|2}}+\cdots+\frac{x^{n-1}}{\underline{|n-1}}+\cdots.$

20. $\frac{1}{\sqrt{1-x}}=1+\frac{1}{2}x+\frac{1\cdot 3}{2\cdot 4}x^2+\cdots+\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2\cdot 4\cdot 6\cdots(2n-2)}x^{n-1}+\cdots.$

21. $\log\left(x+\sqrt{1+x^2}\right)=x-\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5}-\cdots$

$$+(-1)^{n+1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2\cdot 4\cdot 6\cdots(2n-2)}\cdot\frac{x^{2n-1}}{2n-1}+\cdots.$$

22. $\frac{1}{a-bx}=\frac{1}{a}+\frac{b}{a^2}x+\frac{b^2}{a^3}x^2+\cdots+\frac{b^{n-1}}{a^n}x^{n-1}+\cdots.$

Expand each of the following functions into a Maclaurin's series, obtaining four terms:

23. $\sec x.$ **25.** $\log(1+\sin x).$ **27.** $e^{\sin^{-1}x}.$

24. $e^x\sec x.$ **26.** $e^{\sin x}.$ **28.** $\log\cos x.$

Expand each of the following functions into a Fourier's series for values of x between $-\pi$ and π:

29. $x^2.$ **30.** $e^{ax}.$

31. $f(x)$, where $f(x)=-\pi$ if $-\pi<x<0$, and $f(x)=\pi$ if $0<x<\pi$.

32. $f(x)$, where $f(x)=-x$ if $-\pi<x<0$, and $f(x)=0$ if $0<x<\pi$.

33. $f(x)$, where $f(x)=-\pi$ if $-\pi<x<0$, and $f(x)=x$ if $0<x<\pi$.

34. $f(x)$, where $f(x)=0$ if $-\pi<x<0$, and $f(x)=x^2$ if $0<x<\pi$.

Find the limit approached by each of the following functions, as the variable approaches the given value:

35. $\dfrac{\cos x - \cos a}{x - a}, \quad x \doteq a.$

36. $\dfrac{e^x - e^{-x}}{\sin 2x}, \quad x \doteq 0.$

37. $\dfrac{e^{2x} - e^{-2x} - 2x}{x - \sin 2x}, \quad x \doteq 0.$

38. $\dfrac{a^x - b^x}{x}, \quad x \doteq 0.$

39. $\dfrac{(x - a)^3}{e^x - e^a}, \quad x \doteq a.$

40. $\dfrac{\log \cos 2x}{(\pi - x)^2}, \quad x \doteq \pi.$

41. $\dfrac{x - \sin^{-1}x}{\sin^3 x}, \quad x \doteq 0.$

42. $\dfrac{\tan x - x}{x - \sin x}, \quad x \doteq 0.$

43. $\dfrac{\sec x}{\sec 5x}, \quad x \doteq \dfrac{\pi}{2}.$

44. $\dfrac{1 - \log x}{e^{\frac{1}{x}}}, \quad x \doteq 0.$

45. $\dfrac{\log\left(x - \dfrac{\pi}{2}\right)}{\tan x}, \quad x \doteq \dfrac{\pi}{2}.$

46. $\dfrac{\log x}{x^3}, \quad x = \infty.$

47. $\dfrac{x^n}{e^x}, \quad x = \infty.$

48. $\dfrac{\log x}{\csc x}, \quad x \doteq 0.$

49. $\dfrac{a_0x^n + a_1x^{n-1} + \cdots + a_n}{b_0x^n + b_1x^{n-1} + \cdots + b_n}, \quad x = \infty.$

50. $(\pi - x)\tan\dfrac{x}{2}, \quad x \doteq \pi.$

51. $\sec 5x \cos 7x, \quad x \doteq \dfrac{\pi}{2}.$

52. $e^{-x}\log x, \quad x = \infty.$

53. $\dfrac{1}{x - \pi} - \dfrac{1}{\sin x}, \quad x \doteq \pi.$

54. $\dfrac{1}{\log x} - \dfrac{x}{\log x}, \quad x \doteq 1.$

55. $\dfrac{x}{x - 1} - \dfrac{1}{\log x}, \quad x \doteq 1.$

56. $\dfrac{3}{x^3 - 1} - \dfrac{1}{x - 1}, \quad x \doteq 1.$

57. $x^x, \quad x \doteq 0.$

58. $(\tan x)^{\sin x}, \quad x \doteq 0.$

59. $\left(\dfrac{1}{x}\right)^{\tan x}, \quad x \doteq 0.$

60. $x^{\frac{1}{x}}, \quad x = \infty.$

61. $(1 + ax)^{\frac{b}{x}}, \quad x = \infty.$

62. $x^{\frac{1}{1-x}}, \quad x \doteq 1.$

63. $(\cos x)^{\cot x}, \quad x \doteq 0.$

64. $(x + \cos x)^{\frac{1}{x}}, \quad x \doteq 0.$

65. $(1 + ax)^{\frac{b}{x}}, \quad x \doteq 0.$

66. $(\sin x)^{\tan x}, \quad x \doteq \dfrac{\pi}{2}.$

CHAPTER XVI

COMPLEX NUMBERS

165. Graphical representation. A *complex number* is a number of the form $x+iy$, where x and y are real numbers and i is defined by the equation $i^2=-1$ (I, § 12). In this chapter we shall denote a complex number by the letter z, thus

$$z=x+iy.$$

The number x is the *real part* and the number iy is the *imaginary part* of z. When $y=0$, z becomes a real number; and when $x=0$, z becomes a pure imaginary number. When both $x=0$ and $y=0$, then $z=0$; and $z=0$ when, and only when, $x=0$ and $y=0$.

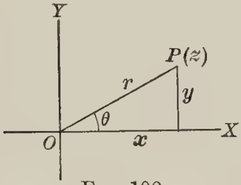

Fig. 109

To obtain a graphical representation of a complex number, construct two axes of coördinates OX and OY (fig. 109), take any point P, and draw OP. Then the complex number $x+iy$ is said to be represented either by the *point* P or by the *vector** OP. For if a complex number, $z=x+iy$, is known, x and y are known, and there corresponds one and only one point P and one and only one vector OP. Conversely, to a point P or a vector OP corresponds one and only one pair of values of x and y, and therefore one and only one complex number. In this connection the axis of x is called the *axis of reals*, since real numbers are represented by points upon it; and the axis of y is called the *axis of imaginaries*, since pure imaginary numbers are represented by points upon it.

If we take O as the origin and OX as the initial line of a system of polar coördinates, we have $x=r\cos\theta$, $y=r\sin\theta$,

and therefore $$z=x+iy=r(\cos\theta+i\sin\theta).$$

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*A *vector* is a straight line fixed in length and direction but not necessarily in position.

The number $r=\sqrt{x^2+y^2}$, which is always taken positive, is called the *modulus* of the complex number and is equal to the length of the vector OP. The angle $\theta=\tan^{-1}\frac{y}{x}$ is called the *argument*, or *amplitude*, of the complex number, and is the angle made with OX by the vector OP. Any multiple of 2π may be added to the argument without altering the complex number, since

$$r[\cos(\theta+2k\pi)+i\sin(\theta+2k\pi)]=r(\cos\theta+i\sin\theta).$$

The modulus is also called the *absolute value* of the complex number and is denoted by $|z|$, thus:

$$|z|=|x+iy|=\sqrt{x^2+y^2}=r.$$

166. Addition and subtraction. If $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$, then, by definition,

$$z_1+z_2=(x_1+x_2)+i(y_1+y_2).$$

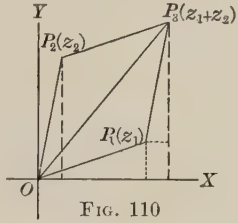

FIG. 110

In fig. 110 let P_1 and P_2 represent the two complex numbers, and let OP_1 and OP_2 be the corresponding vectors. Construct the parallelogram $OP_1P_3P_2$. Then it is easy to see that P_3 has the coördinates (x_1+x_2, y_1+y_2) and therefore represents the complex number z_1+z_2. The addition of complex numbers is thus seen to be analogous to the composition of forces or velocities.

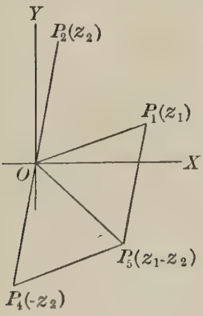

FIG. 111

Since $OP_1=|z_1|$, $OP_2=|z_2|$, and $OP_3=|z_1+z_2|$,

$$|z_1+z_2|\leqq|z_1|+|z_2|,$$

the equality sign holding only when OP_1 and OP_2 have the same direction.

To subtract z_2 from z_1, we first change the sign of z_2 and add the result to z_1. Graphically, the change of sign of z_2 corresponds to replacing P_2 (fig. 111) by P_4, symmetrical to P_2 with respect to O, or to turning the vector OP_2 through an angle of 180°. The parallelogram $OP_1P_5P_4$ is then completed, giving the point P_5 corresponding to z_1-z_2.

167. Multiplication and division. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then, by definition,

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

If we use polar coördinates, we have $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, and

$$\begin{aligned} z_1 z_2 &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) \\ &\qquad + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]. \end{aligned}$$

Hence, in the multiplication of two complex numbers, the moduli are multiplied together and the arguments are added. Graphically, the vector corresponding to a product is found by rotating the vector of the multiplicand in the positive direction through an angle equal to the argument of the multiplier, and multiplying the length of the vector of the multiplicand by the modulus of the multiplier. In particular, the multiplication by i is represented by rotating a vector through an angle of 90°; and the multiplication by -1 is represented by rotating a vector through an angle of 180°, as noted in § 166.

The quotient $\frac{z_1}{z_2}$ is a number which multiplied by z_2 gives z_1. Hence, if $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)].$$

Graphically, the vector corresponding to a quotient is found by rotating the vector of the dividend in the negative direction through an angle equal to the argument of the divisor, and dividing the length of the vector of the dividend by the modulus of the divisor.

168. Involution and evolution. The value of z^n, where n is a positive integer, is obtained by successive multiplication of z by itself. Therefore, if $z = r(\cos\theta + i\sin\theta)$,

$$z^n = r^n(\cos n\theta + i\sin n\theta). \qquad (1)$$

The root $z^{\frac{1}{n}}$, where n is a positive integer, is a number which raised to the nth power gives z. Accordingly we have, at first sight, $z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\frac{\theta}{n} + i \sin\frac{\theta}{n}\right)$. But if we remember (§ 165) that $z = r[\cos(\theta + 2k\pi) + i \sin(\theta + 2k\pi)]$, where k is zero or an integer, we have also

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i \sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right]. \qquad (2)$$

There are here n distinct values of $z^{\frac{1}{n}}$, obtained by giving to k the n values $0, 1, 2, \cdots, (n-1)$. Since the equation $z^n = c$ has only n roots (I, § 42), (2) gives all the values of z. In this result $r^{\frac{1}{n}}$ means the positive numerical root of the number r, such as may be found by use of a table of logarithms.

By a combination of (1) and (2), we have

$$z^{\frac{p}{q}} = r^{\frac{p}{q}}\left[\cos\left(\frac{p\theta}{q} + \frac{2kp\pi}{q}\right) + i \sin\left(\frac{p\theta}{q} + \frac{2kp\pi}{q}\right)\right], \qquad (3)$$

where $k = 0, 1, 2, \cdots, (q-1)$.

Finally,
$$z^{-m} = \frac{1}{z^m} = \frac{\cos 0 + i \sin 0}{r^m(\cos m\theta + i \sin m\theta)}$$
$$= r^{-m}[\cos(-m\theta) + i \sin(-m\theta)].$$

The *nth roots of unity* can be found by placing $r = 1$ and $\theta = 0$ in (2). Then

$$\sqrt[n]{1} = \cos\frac{2k\pi}{n} + i \sin\frac{2k\pi}{n}.$$

The points which represent these roots graphically are the vertices of a regular polygon of n sides, inscribed in a circle with center at O and radius unity, the first vertex lying on the axis of reals.

Similarly, $\sqrt[n]{-1} = \cos\frac{(2k+1)\pi}{n} + i \sin\frac{(2k+1)\pi}{n}$.

169. Exponential and trigonometric functions. The exponential and the trigonometric functions are defined in elementary work in a manner which assumes that the variable is real. For example, the definition of $\sin x$ requires the construction of a real triangle with an angle equal to x, and the definition of e^x involves actual involution and evolution. In order to be able to regard the independent variable as a complex number, we adopt the following *definitions:*

$$e^z = 1 + \frac{z}{1} + \frac{z^2}{\underline{|2}} + \frac{z^3}{\underline{|3}} + \cdots, \tag{1}$$

$$\sin z = z - \frac{z^3}{\underline{|3}} + \frac{z^5}{\underline{|5}} - \frac{z^7}{\underline{|7}} + \cdots, \tag{2}$$

$$\cos z = 1 - \frac{z^2}{\underline{|2}} + \frac{z^4}{\underline{|4}} - \frac{z^6}{\underline{|6}} + \cdots. \tag{3}$$

When z is real, each of these functions is the corresponding real function, since these series are those found in § 31.

When z is complex, each of the series converges. To show this, place $z = r(\cos\theta + i\sin\theta)$ in (1), for example. We have

$$\left(1 + r\cos\theta + \frac{r^2}{\underline{|2}}\cos 2\theta + \frac{r^3}{\underline{|3}}\cos 3\theta + \cdots\right)$$
$$+ i\left(r\sin\theta + \frac{r^2}{\underline{|2}}\sin 2\theta + \frac{r^3}{\underline{|3}}\sin 3\theta + \cdots\right).$$

Each term of the two series in parentheses is less in absolute value than a corresponding term of the known convergent series $1 + r + \frac{r^2}{\underline{|2}} + \frac{r^3}{\underline{|3}} + \cdots$. Therefore each of these series converges (§ 153) and hence (1) converges. In the same manner it may be shown that (2) and (3) converge. It follows that each of the series (1), (2), and (3) may properly be used to define a function. It is necessary, however, to give new proofs of the properties of the functions, based upon the new definitions.

From (1) we have

$$e^0 = 1, \qquad e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2},$$

which are the fundamental properties of the exponential function.

From (1), also, if we replace z by iz, we have

$$e^{iz} = 1 + \frac{iz}{1} + \frac{(iz)^2}{\underline{|2}} + \frac{(iz)^3}{\underline{|3}} + \frac{(iz)^4}{\underline{|4}} + \frac{(iz)^5}{\underline{|5}} + \cdots$$
$$= \left(1 - \frac{z^2}{\underline{|2}} + \frac{z^4}{\underline{|4}} - \cdots\right) + i\left(z - \frac{z^3}{\underline{|3}} + \frac{z^5}{\underline{|5}} - \cdots\right).$$

From this, with reference to (2) and (3), we obtain the formula

$$e^{iz} = \cos z + i \sin z, \tag{4}$$

which establishes a relation between the exponential and the trigonometric functions.

By changing the sign of z in (4) we have

$$e^{-iz} = \cos z - i \sin z,$$

whence

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \tag{5}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}. \tag{6}$$

From (4), (5), and (6) it is easily shown that the trigonometric formulas obtained in elementary work for real variables are true for complex variables also (see Exs. 1 and 2).

We shall use these relations to separate the functions (1), (2), and (3) into real and imaginary parts, as follows:

$$e^{x+iy} = e^x e^{iy} = e^x \cos y + ie^x \sin y.$$

$$\begin{aligned}\sin(x+iy) &= \frac{1}{2i}\left(e^{ix-y} - e^{-ix+y}\right)\\ &= \frac{1}{2i}\left[e^{-y}(\cos x + i \sin x) - e^y(\cos x - i \sin x)\right]\\ &= \frac{e^y + e^{-y}}{2} \sin x + i\,\frac{e^y - e^{-y}}{2} \cos x\\ &= \cosh y \sin x + i \sinh y \cos x.\end{aligned}$$

Similarly,

$$\begin{aligned}\cos(x+iy) &= \frac{e^y + e^{-y}}{2} \cos x - i\,\frac{e^y - e^{-y}}{2} \sin x\\ &= \cosh y \cos x - i \sinh y \sin x.\end{aligned}$$

If $x = 0$, we have

$$\sin iy = i\,\frac{e^y - e^{-y}}{2} = i \sinh y,$$

$$\cos iy = \frac{e^y + e^{-y}}{2} = \cosh y.$$

Ex. 1. Prove $\sin^2 z + \cos^2 z = 1$.

From (5) and (6),

$$\sin^2 z + \cos^2 z = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 + \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 = 1.$$

Ex. 2. Prove $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$.

From (5),

$$\begin{aligned}\sin(z_1 + z_2) &= \frac{e^{iz_1 + iz_2} - e^{-iz_1 - iz_2}}{2i}\\ &= \frac{1}{2i}[e^{iz_1}e^{iz_2} - e^{-iz_1}e^{-iz_2}]\\ &= \frac{1}{2i}[(\cos z_1 + i \sin z_1)(\cos z_2 + i \sin z_2)\\ &\qquad - (\cos z_1 - i \sin z_1)(\cos z_2 - i \sin z_2)]\\ &= \sin z_1 \cos z_2 + \cos z_1 \sin z_2.\end{aligned}$$

170. The logarithmic function. If $z = e^w$, then, by definition,

$$w = \log z.$$

The properties of the logarithmic function, namely

$$\log(z_1 z_2) = \log z_1 + \log z_2, \qquad \log\frac{z_1}{z_2} = \log z_1 - \log z_2,$$

$$\log z^n = n \log z, \qquad \log 1 = 0, \qquad \log 0 = -\infty,$$

are deduced from the definition, as in the case of real variables.

The logarithm of a complex number is itself a complex number. For, let us place

$$z = x + iy = r(\cos\theta + i \sin\theta) = re^{i\theta}.$$

Then

$$\log z = \log r + \log e^{i\theta} = \log r + i\theta = \frac{1}{2}\log(x^2 + y^2) + i \tan^{-1}\frac{y}{x}.$$

Here $\log r$ is the logarithm of the positive number r, and may be found as in I, § 154.

We may now find the logarithm of a real negative number. For, if $-a$ is such a number, we may write $-a = a(\cos\pi + i \sin\pi) = ae^{i\pi}$, whence

$$\log(-a) = \log a + i\pi.$$

In particular, $\log(-1) = i\pi$.

It is to be noticed that, in the domain of the complex numbers, a logarithm is not a unique quantity. For

$$e^w = e^{w+2ki\pi} = z,$$

where k is zero or an integer. Therefore

$$\log z = w + 2\,ki\pi.$$

We may express this as follows: *The exponential function is a periodic function with the period $2\,i\pi$, and the logarithm has an infinite number of values, differing by multiples of $2\,i\pi$.*

171. Functions of a complex variable in general. We have seen that functions of a complex variable obtained by operating on $x + iy$ with the fundamental operations of algebra, or involving the elementary transcendental functions, are themselves complex numbers of the form $u + iv$, where u and v are real functions of x and y. Let us now assume the expression $w = u + iv$, and inquire what conditions it must satisfy in order that it may be a function of $z = x + iy$.

In the first place, it is to be noticed that in the broadest sense of the word "function" (I, § 20), w is always a function of z, since when z is given, x and y are determined and therefore u and v are determined. But this definition is too broad for our present purpose, and we shall restrict it by demanding that the function shall have a definite derivative for a definite value of z. The force of this restriction is seen as follows: In order to obtain an increment of z, we may assign at pleasure increments Δx and Δy to x and y respectively and obtain $\Delta z = \Delta x + i\Delta y$. The direction in which the point Q (fig. 112), which corresponds to $z + \Delta z$ in the graphical representation, lies from P, which corresponds to z, depends then on the ratio $\frac{\Delta y}{\Delta x}$, which may have any value whatever. Corresponding to a given increment Δz, w takes an increment Δw, where (§ 110)

$$\Delta w = \left(\frac{\partial u}{\partial x} + \epsilon_1\right)\Delta x + \left(\frac{\partial u}{\partial y} + \epsilon_2\right)\Delta y + i\left[\left(\frac{\partial v}{\partial x} + \epsilon_3\right)\Delta x + \left(\frac{\partial v}{\partial y} + \epsilon_4\right)\Delta y\right].$$

Dividing by $\Delta z = \Delta x + i\Delta y$ and taking the limit as $\Delta x \doteq 0$ and $\Delta y \doteq 0$, we have

$$\operatorname*{Lim}_{\Delta z \doteq 0} \frac{\Delta w}{\Delta z} = \frac{\dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} + \left(\dfrac{\partial u}{\partial y} + i\dfrac{\partial v}{\partial y}\right)\dfrac{dy}{dx}}{1 + i\dfrac{dy}{dx}}. \tag{1}$$

Unless special conditions are imposed upon u and v, the expression on the right-hand side of the above equation involves $\frac{dy}{dx}$, and the value of $\operatorname*{Lim}_{\Delta z \doteq 0} \frac{\Delta w}{\Delta z}$ depends upon the direction in which the point Q (fig. 112) approaches the point P. Now the value of the right-hand side of (1), when $\frac{dy}{dx} = 0$, is $\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$, and its value when $\frac{dy}{dx} = \infty$ is $\dfrac{\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}}{i}$. Equating these two values, we have

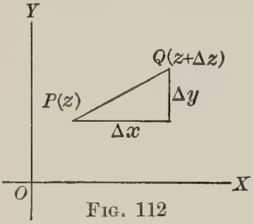

FIG. 112

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0. \tag{2}$$

This, then, is the necessary condition that $\operatorname*{Lim}_{\Delta z \doteq 0} \frac{\Delta w}{\Delta z}$ should be the same for the two values $\frac{dy}{dx} = 0$ and $\frac{dy}{dx} = \infty$. It is also the sufficient condition that $\operatorname*{Lim}_{\Delta z \doteq 0} \frac{\Delta w}{\Delta z}$ should be the same for all values of $\frac{dy}{dx}$, for if (1) is simplified by aid of (2), $\frac{dy}{dx}$ disappears from it.

Now (2) is equivalent to the two conditions

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}. \end{aligned} \tag{3}$$

Hence *the equations* (3) *are the necessary and sufficient conditions that the function* $u + iv$ *should have a derivative with respect to* $x + iy$ *which depends upon the value of* $x + iy$ *only.*

A function $u + iv$ which satisfies conditions (3) is called an *analytic function* of $x + iy$.

172. Conjugate functions. Two real functions u and v, which satisfy conditions (3), § 171, are called *conjugate functions.* By differentiating the first equation of (3), § 171, with respect to x, the second with respect to y, and adding the results, we have

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

Also, by differentiating the first equation of (3), § 171, with respect to y, the second with respect to x, and taking the difference of the results, we have

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0.$$

That is, *each of a pair of conjugate functions is a solution of the differential equation*

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=0.$$

Let us construct now the two families (§ 173) of curves $u=c_1$ and $v=c_2$. If (x_1, y_1) is a point of intersection of two of these curves, one from each family, the equations of the tangent lines at (x_1, y_1) are (§ 115, Ex. 1)

$$(x-x_1)\left(\frac{\partial u}{\partial x}\right)_1+(y-y_1)\left(\frac{\partial u}{\partial y}\right)_1=0,$$

$$(x-x_1)\left(\frac{\partial v}{\partial x}\right)_1+(y-y_1)\left(\frac{\partial v}{\partial y}\right)_1=0.$$

But from (3), § 171,

$$\left(\frac{\partial u}{\partial x}\right)_1\left(\frac{\partial v}{\partial x}\right)_1+\left(\frac{\partial u}{\partial y}\right)_1\left(\frac{\partial v}{\partial y}\right)_1=0.$$

Hence the two curves intersect at right angles; that is, every curve of one family intersects every curve of the other family at right angles. We express this by saying that the families of curves corresponding to two conjugate functions form an orthogonal system.

Examples of conjugate functions and of orthogonal systems of curves may be found by taking the real and the imaginary parts of any function of a complex variable.

Ex. 1. $(x + iy)^2$
$= x^2 - y^2 + 2\,ixy.$

From this it follows that $x^2 - y^2$ and $2\,xy$ are conjugate functions, and that the curves $x^2 - y^2 = c_1$ and $xy = c_2$ form an orthogonal system (fig. 113). In fact, the system consists of two families of rectangular hyperbolas, the one having the coördinate axes for axes of the curves and the other having the coördinate axes for asymptotes.

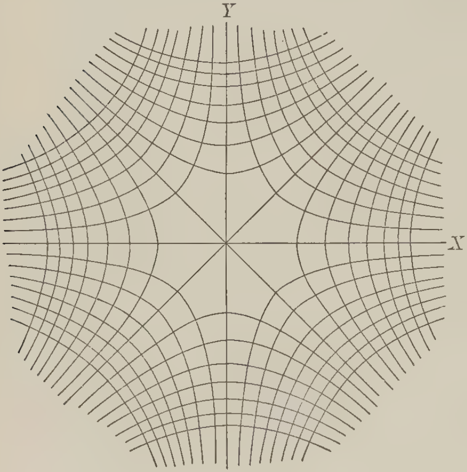

Fig. 113

Ex. 2. $\log(x + iy)$
$= \log\sqrt{x^2 + y^2} + i \tan^{-1}\frac{y}{x}.$

Hence $\log\sqrt{x^2 + y^2}$ and $\tan^{-1}\frac{y}{x}$ are conjugate functions, and the curves $x^2 + y^2 = c_1$ and $y = c_2 x$ form an orthogonal system (fig. 114). In fact, one family of curves consists of circles with their centers at the origin, and the other consists of straight lines through the origin.

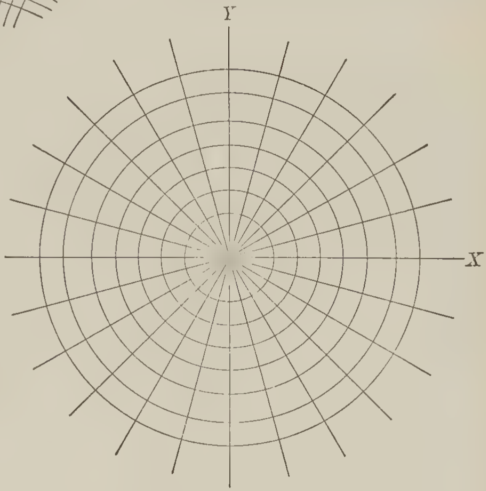

Fig. 114

PROBLEMS

Find the sums of the following pairs of complex numbers and the difference obtained by subtracting the second from the first, and express the results graphically:

1. $3 + 2\,i,\ 4 + 5\,i.$

2. $-3 - 7\,i,\ 4 + 8\,i.$

3. $6 - 10\,i,\ 3 + 2\,i.$

4. $-8 + 12\,i,\ 6 + 9\,i.$

Find the products of the following pairs of complex numbers and the quotient of the first by the second, and express the results graphically:

5. $\cos\frac{\pi}{6}+i\sin\frac{\pi}{6},\ \cos\frac{\pi}{4}+i\sin\frac{\pi}{4}$.

6. $\cos\frac{\pi}{3}+i\sin\frac{\pi}{3},\ \cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$.

7. $1+i\sqrt{3},\ \sqrt{3}+i$.

8. $2+2i\sqrt{3},\ 3-3i\sqrt{2}$.

9. $1+i,\ -1+i$.

10. $1+i,\ 1-i$.

Find the following powers and express the results graphically:

11. $(2+3i)^2$. **12.** $(1-i\sqrt{3})^2$. **13.** $(1+i)^3$. **14.** $(1+i)^4$.

Find the following roots and locate them graphically:

15. $\sqrt[4]{1}$. **16.** $\sqrt[4]{-1}$. **17.** $\sqrt[5]{1}$. **18.** $\sqrt[5]{-1}$. **19.** $\sqrt[3]{-8}$. **20.** $\sqrt[3]{8}$.

21. Prove, for complex numbers, $\cos(z_1+z_2)=\cos z_1\cos z_2-\sin z_1\sin z_2$.

22. Prove, for complex numbers, $\sin z_1+\sin z_2=2\sin\frac{1}{2}(z_1+z_2)\cos\frac{1}{2}(z_1-z_2)$.

23. Prove $\sinh iy=i\sin y$. **24.** Prove $\cosh iy=\cos y$.

Find the values of the following logarithms:

25. $\log(-2)$. **26.** $\log(1+i)$. **27.** $\log(-1+i)$. **28.** $\log i$.

Find the orthogonal systems of curves defined by the real and the imaginary parts of the following functions:

29. $\frac{1}{z}$. **30.** $\log\frac{z-a}{z+a}$. **31.** $\log(z^2-a^2)$.

Find the orthogonal systems of curves defined by the real and the imaginary parts of the following functions, using polar coördinates:

32. z^3. **33.** $\sqrt{z}$.

CHAPTER XVII

DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

173. Introduction. Consider the equation

$$f(x, y, c) = 0, \tag{1}$$

in which c is an arbitrary constant. If c is given a fixed value, (1) is the equation of a certain curve; and if c is supposed to take all values, the totality of the curves thus represented by (1) is called a *family of curves.*

To determine the curves of the family which pass through any fixed point $P(x_1, y_1)$, we may substitute x_1 and y_1 for x and y in (1), thereby forming the equation

$$f(x_1, y_1, c) = 0. \tag{2}$$

The number of roots of (2), regarded as an equation in c, is the number of curves of family (1) which pass through P_1, and their equations are found by substituting these values of c in (1).

The direction of any curve of family (1) is given by the equation (§ 115, (1))

$$\frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}, \tag{3}$$

which, in general, involves c. In general, however, we may eliminate c from (1) and (3), the result being an equation of the form

$$F\left(x, y, \frac{dy}{dx}\right) = 0. \tag{4}$$

If we substitute the coördinates of P_1 in (4), the values of $\frac{dy}{dx}$ which satisfy the resulting equation are the slopes of the respective curves of (1) which pass through P_1. Hence (4) defines the same family of curves that is defined by (1), but by means of the directions of the curves instead of the explicit equations of the curves themselves. Hence (4) *is called the differential equation of the family of curves represented by* (1).

We have seen how an equation of type (1) leads to a differential equation of type (4). Conversely, *to an equation of form* (4) *there always corresponds a family of curves which may be represented by an equation of form* (1). For if the coördinates of a point P_1 (fig. 115) are assigned to x and y in (4), (4) determines one or more directions through P_1. Following one of these directions, we may move to a second point P_2. If the coördinates of P_2 are substituted in (4), a direction is determined in which we may move to a third point P_3. Proceeding in this way, we trace a broken line such that the coördinates of every vertex and the direction of the following segment at that point satisfy (4). The limit of this broken line, as the length of each segment approaches the limit zero, is a curve such that the coördinates of each point and the direction of the curve at that point satisfy (4). Since in this construction P_1 may be any point of the plane, there is evidently a family of curves represented by (4), as we set out to prove. The constant c in the equation of the family may be taken, for example, as the ordinate of the point in which a curve of the family cuts the axis of y or any other line $x = x_1$. Hence *every differential equation of form* (4) *has a solution of form* (1).

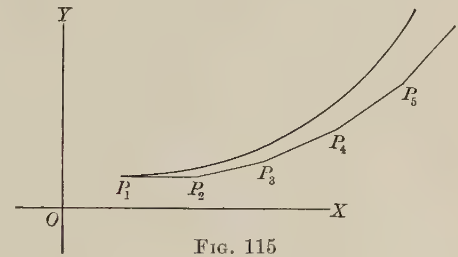

FIG. 115

The problem of proceeding from a differential equation (4) to its solution (1) is, however, a difficult one, which can be solved only in the simpler cases. Some of these cases have been discussed in this volume for equations in which $\frac{dy}{dx}$ appears to the first power only. These are the following:

I. Variables separable. (§ 77)

Ia. Homogeneous equation. (§ 78)

Ib. Equation of the form

$$(a_1x + b_1y + c_1)\,dx + (a_2x + b_2y + c_2)\,dy = 0. \qquad (\S\ 79)$$

II. The linear equation (§ 80)
IIa. Bernouilli's equation. (§ 81)
III. The exact equation. (§ 147)
IIIa. Solution by integrating factors. (§ 149)

While the above methods often prove serviceable, the student should appreciate that they may all fail with a given differential equation, since the above list does not contain all possible differential equations which are of the first degree in $\frac{dy}{dx}\cdot$ Moreover, the solution will in general involve integrations of which the results can be expressed as elementary functions only in the simpler cases.

174. Solution by series. The solution of a differential equation may usually be expressed in the form of a power series.

Ex. $\frac{dy}{dx} = x^2 + y^2$.

Assume $y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$. Substituting in the given equation, we have

$$a_1 + 2\,a_2x + 3\,a_3x^2 + \cdots = x^2 + (a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots)^2,$$

in which the coefficients of like powers of x on the two sides of the equation must be equal, since the equation is true for all values of x.

Equating coefficients, we have

$$\begin{aligned} a_1 &= a_0^2, \\ 2\,a_2 &= 2\,a_0a_1, \\ 3\,a_3 &= 1 + a_1^2 + 2\,a_0a_2, \\ &\cdots\cdots\cdots \end{aligned}$$

whence

$$\begin{aligned} a_1 &= a_0^2, \\ a_2 &= a_0^3, \\ a_3 &= \tfrac{1}{3}(1 + 3\,a_0^4), \\ &\cdots\cdots\cdots \end{aligned}$$

Hence the required solution is

$$y = a_0 + a_0^2x + a_0^3x^2 + \tfrac{1}{3}(1 + 3\,a_0^4)\,x^3 + \cdots.$$

175. Equations not of the first degree in the derivative. If the differential equation of the first order is of higher degree than the first in $\frac{dy}{dx}$, new methods of solution are necessary. Denoting $\frac{dy}{dx}$ by p, we shall make three cases:

1. Equations solvable for p.
2. Equations solvable for y.
3. Equations solvable for x.

176. Equations solvable for p. Let the given equation be an equation of the nth degree in p, and let the roots of the equation, regarded as an algebraic equation in p, be $p_1, p_2, \cdots, p_n$, where $p_1, p_2, \cdots, p_n$ are functions of x and y, or constants. Then the equation may be written in the form (I, § 42)

$$(p-p_1)(p-p_2)\cdots(p-p_n)=0. \tag{1}$$

But (1) is satisfied when and only when one of the n factors is zero, and hence the solution of (1) is made to depend upon the solutions of the n equations

$$p-p_1=0, \quad p-p_2=0, \quad \cdots, \quad p-p_n=0. \tag{2}$$

Let the solution of $p-p_1=0$ be $f_1(x, y, c_1)=0$, the solution of $p-p_2=0$ be $f_2(x, y, c_2)=0$, etc., where $c_1, c_2, \cdots$ are arbitrary constants. Since each of these constants is arbitrary, however, and there is no necessity of distinguishing among them, we will denote them all by the same letter c.

Now form the equation

$$f_1(x, y, c)\cdot f_2(x, y, c)\cdots f_n(x, y, c)=0. \tag{3}$$

The values of x and y which make any factor of (3) zero satisfy (1) by making the corresponding factor of its left-hand member zero. Hence (3) is a solution of (1), since the values of x and y which satisfy (3) are all the values of x and y necessary to satisfy (1); and since (3) contains an arbitrary constant, it is the general solution.

Ex. 1. $p^3-2p^2+\left(2y-x^2-\dfrac{y^2}{x^2}\right)p-2\left(2y-x^2-\dfrac{y^2}{x^2}\right)=0.$

Solving this equation for p, we have

$$p=2, \qquad p=\frac{y}{x}-x, \quad \text{and} \quad p=-\frac{y}{x}+x.$$

The solution of the first equation is evidently $y-2x+c=0$. The second equation, when written in the form $\dfrac{dy}{dx}-\dfrac{1}{x}y=-x$, is seen to be a linear equation, and its solution is $y+x^2-cx=0$.

The third equation may also be written in the form of a linear equation, and its solution is $x^3-3xy+c=0$.

Hence the solution of the original equation is

$$(y-2x+c)(y+x^2-cx)(x^3-3xy+c)=0.$$

Ex. 2. $y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0.$

Solving for p, we have $\quad p = -\frac{x}{y} \pm \frac{\sqrt{x^2+y^2}}{y}.$

The equation
$$p = -\frac{x}{y} + \frac{\sqrt{x^2+y^2}}{y}$$
is a homogeneous equation, and its solution is $\sqrt{x^2+y^2} - x + c = 0.$

The equation
$$p = -\frac{x}{y} - \frac{\sqrt{x^2+y^2}}{y}$$
is also a homogeneous equation, and its solution is $\sqrt{x^2+y^2} + x - c = 0.$

Hence the required solution is $y^2 + 2cx - c^2 = 0.$

177. Equations solvable for y. Let the given equation be
$$f(x, y, p) = 0. \tag{1}$$

Solving this equation for y, we have the equation
$$y = F(x, p). \tag{2}$$

Differentiating (2) with respect to x, and replacing $\frac{dy}{dx}$ by p, we have the equation
$$p = \phi\left(x, p, \frac{dp}{dx}\right), \tag{3}$$
in which p and x are the variables. Let the solution of (3) be
$$\psi(x, p, c) = 0. \tag{4}$$

Eliminating p between (1) and (4), we have a function of x, y, and an arbitrary constant which is, in general, a solution of (1). But the process of solution may bring in extraneous factors or otherwise lead to error, and the solution should be tested by substitution in (1).

If the elimination cannot be performed, equations (1) and (4) may be taken simultaneously as the parametric representation (I, § 163) of the solution, p being the variable parameter.

Ex. 1. $xp^2 - 2yp + ax = 0.$

Solving for y, we have
$$y = \frac{xp}{2} + \frac{ax}{2p}. \tag{1}$$

By differentiating (1) with respect to x, we obtain
$$p = \frac{1}{2}\left(p + x\frac{dp}{dx}\right) + \frac{a}{2}\left(\frac{1}{p} - \frac{x}{p^2}\frac{dp}{dx}\right),$$
or
$$\left(\frac{a}{p} - p\right)\left(1 - \frac{x}{p}\frac{dp}{dx}\right) = 0. \tag{2}$$

The first factor placed equal to zero gives $p = \pm\sqrt{a}$. If this value is substituted for p in the given equation, we have $2ax \pm 2y\sqrt{a} = 0$, which is found on trial to be a solution of the equation. This solution, however, involves no arbitrary constant, and hence is of a different type from that already considered. It will be discussed in § 182.

Placing the second factor equal to zero, and solving the resulting equation, we find $p = cx$. Substituting this value of p in (1), we have, as the general solution,

$$y = \frac{cx^2}{2} + \frac{a}{2c}.$$

Ex. 2. *Clairaut's equation*, $y = px + f(p)$.

As this equation already expresses y in terms of x and p, we proceed immediately to differentiate with respect to x, with the result

$$[x + f'(p)]\frac{dp}{dx} = 0.$$

As in Ex. 1, placing the first factor equal to zero cannot give us the general solution. Neglecting that factor, we have $\frac{dp}{dx} = 0$, whence $p = c$. Substituting this value for p in the original equation, we have, as the general solution,

$$y = cx + f(c).$$

Hence the general solution of Clairaut's equation may be written down immediately by merely replacing p by c in the given equation. The ease of this solution makes it desirable to solve any equation for y, in the hope that the new equation may be Clairaut's equation.

Ex. 3. $y = px + a\sqrt{1 + p^2}$.

Since the equation is in the form of Clairaut's equation, with $f(p) = a\sqrt{1 + p^2}$, its solution is

$$y = cx + a\sqrt{1 + c^2}.$$

178. Equations solvable for x. If the given equation can be solved for x, with the result

$$x = f(y, p), \tag{1}$$

we may form a new equation,

$$\frac{1}{p} = \phi\left(y, p, \frac{dp}{dy}\right), \tag{2}$$

by differentiating (1) with respect to y, and replacing $\frac{dx}{dy}$ by $\frac{1}{p}$.

Let the solution of (2) be

$$\psi(y, p, c) = 0. \tag{3}$$

Then (1) and (3) may be taken simultaneously as the parametric representation of the solution of (1). Or p may be eliminated from (1) and (3), the result being a function of x, y, and an arbitrary constant, which is, in general, a solution of (1), but which should be tested by substitution in (1).

Ex. $x - 2p - \log p = 0$.

Solving for x, we have $$x = 2p + \log p. \tag{1}$$

Differentiating with respect to y, we obtain the equation

$$dy = (2p + 1)\,dp, \tag{2}$$

the solution of which is $$y = p^2 + p + c. \tag{3}$$

Since the result of eliminating p from (1) and (3) is complicated, we take (1) and (3) as the parametric representation of the solution of (1).

179. Envelopes. Let $$f(x, y, c) = 0 \tag{1}$$

be the equation of a family of curves formed by giving different values to the arbitrary parameter c. If any particular value of c is increased by Δc, the equation of the corresponding curve is

$$f(x, y, c + \Delta c) = 0. \tag{2}$$

The limiting positions of the points of intersection of (1) and (2), as $\Delta c \doteq 0$, will be called limit points on (1). We wish to discuss the locus of the limit points.

One method is evidently to solve (1) and (2) simultaneously for x and y in terms of c and Δc. The limiting values of x and y, as $\Delta c \doteq 0$, will be the coördinates of a limit point expressed in terms of c. If c is eliminated from these values of x and y, the result is the Cartesian equation of the locus of the limit points.

A second method is as follows: Any point of intersection of (1) and (2) is a point of

$$\frac{f(x, y, c + \Delta c) - f(x, y, c)}{\Delta c} = 0, \tag{3}$$

so that we may use (3) in place of (2). As it is only the limiting positions of the points of intersection of (1) and (3) that are to be considered, we may take the limit of (3) as $\Delta c \doteq 0$, i.e.

$$\frac{\partial f}{\partial c} = 0. \tag{4}$$

Then (4) is a curve passing through the limit points. Eliminating c between (1) and (4), we obtain the equation of the required locus.

Ex. Find the locus of the limit points on the straight lines represented by $y - mx - a\sqrt{1+m^2} = 0$, m being the variable parameter.

First method. We first solve the equations

$$y - mx - a\sqrt{1+m^2} = 0, \tag{1}$$

$$y - (m+\Delta m)x - a\sqrt{1+(m+\Delta m)^2} = 0, \tag{2}$$

with the results $$x = a\frac{\sqrt{1+m^2} - \sqrt{1+(m+\Delta m)^2}}{\Delta m}, \tag{3}$$

$$y = am\frac{\sqrt{1+m^2} - \sqrt{1+(m+\Delta m)^2}}{\Delta m} + a\sqrt{1+m^2}. \tag{4}$$

Taking the limits of (3) and (4), as $\Delta m \doteq 0$, we have

$$x = -a\frac{d}{dm}\sqrt{1+m^2} = -\frac{am}{\sqrt{1+m^2}}, \tag{5}$$

$$y = \frac{a}{\sqrt{1+m^2}}, \tag{6}$$

the coördinates of any limit point expressed in terms of m.

Eliminating m, we have

$$x^2 + y^2 = a^2. \tag{7}$$

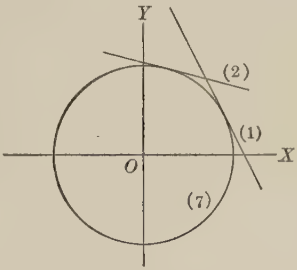

FIG. 116

It is thus evident (fig. 116) that the locus is a circle tangent to each of the straight lines represented by the given equation.

Second method. From (1),

$$\frac{\partial f}{\partial m} = -x - \frac{am}{\sqrt{1+m^2}} = 0. \tag{8}$$

Eliminating m from (1) and (8), we have

$$x^2 + y^2 - a^2 = 0,$$

the locus of which is the circle found by the first method.

180. In the illustrative example of the last article, the locus of the limit points of the family, as those curves approach coincidence, is a curve tangent to every curve of the family. Hence the question is suggested, Is the locus of limit points always tangent to every curve of the family? To answer this question, we proceed as follows:

Let (x_1, y_1) be a limit point on one of the curves represented by

$$f(x, y, c) = 0. \tag{1}$$

Then its coördinates satisfy (1) and

$$\frac{\partial f}{\partial c} = 0. \tag{2}$$

The tangent to the curve of the family at (x_1, y_1) is

$$(x - x_1)\left(\frac{\partial f}{\partial x}\right)_{yc} + (y - y_1)\left(\frac{\partial f}{\partial y}\right)_{xc} = 0, \tag{3}$$

where the meaning of $\left(\frac{\partial f}{\partial x}\right)_{yc}$ is as in Ex. 3, § 113.

The equation of the locus of the limit points may be found theoretically by substituting the value of c in terms of x and y from (2) in (1). Then the equation of the tangent to the locus of the limit points at (x_1, y_1) is

$$(x - x_1)\left(\frac{\partial f}{\partial x}\right)_{y} + (y - y_1)\left(\frac{\partial f}{\partial y}\right)_{x} = 0, \tag{4}$$

or (§ 113, Ex. 3)

$$(x - x_1)\left[\left(\frac{\partial f}{\partial x}\right)_{yc} + \frac{\partial f}{\partial c}\frac{\partial c}{\partial x}\right] + (y - y_1)\left[\left(\frac{\partial f}{\partial y}\right)_{xc} + \frac{\partial f}{\partial c}\frac{\partial c}{\partial y}\right] = 0. \tag{5}$$

But since $\frac{\partial f}{\partial c} = 0$, (5) reduces to

$$(x - x_1)\left(\frac{\partial f}{\partial x}\right)_{yc} + (y - y_1)\left(\frac{\partial f}{\partial y}\right)_{xc} = 0, \tag{6}$$

which is the same as (3). Hence, in general, the locus of the limit points is tangent to every curve of the family.

There may be limit points, however, which lie on a locus that is not tangent to every curve of the family. For let each curve of the family have one or more *singular points*, i.e. points for which $\left(\frac{\partial f}{\partial x}\right)_y = 0$, $\left(\frac{\partial f}{\partial y}\right)_x = 0$. Then such points will be a part of the locus of the limit points; for, from (1), we have

$$\left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy + \frac{\partial f}{\partial c} dc = 0.$$

But at a singular point the first terms vanish, and hence the coördinates of any singular point satisfy $f(x, y, c) = 0$ and $\frac{\partial f}{\partial c} = 0$, and are limit points. But at a singular point the equation of the tangent becomes indeterminate, and hence the locus of the limit points may or may not be tangent to each curve of the family. Accordingly, we shall separate that part of the locus of limit points which is tangent to each curve of the family, and give it the special name *envelope*. That is, *the envelope** *is that part of the locus of the limit points of a family of curves which is tangent to every curve of the family.* Hence, in finding the equation of the envelope, it is necessary to find the locus of the limit points, throwing out any extraneous factor brought in by the elimination, and also discarding any part of the locus which is not tangent to each curve of the family.

181. The second method of finding the locus of limit points is exactly the method of determining the condition that $f(x, y, c) = 0$, if it is an algebraic equation in c, shall have equal roots (I, § 64). Hence, if we form the discriminant of $f(x, y, c) = 0$, regarded as an equation in c, and place it equal to zero, the locus of the resulting equation will contain the envelope. If there are any additional loci, they are the loci of singular points, or correspond to extraneous factors brought in by the elimination.

Ex. The equation of the example in § 179 may be written in the form

$$(x^2 - a^2)m^2 - 2xym + (y^2 - a^2) = 0.$$

The discriminant of this quadratic equation in m is (I, § 37)

$$(-2xy)^2 - 4(x^2 - a^2)(y^2 - a^2) = 4a^2(x^2 + y^2 - a^2).$$

Hence the condition for equal roots is

$$x^2 + y^2 - a^2 = 0,$$

and this is the equation of the envelope, since there are no extraneous factors.

182. Singular solutions. Let

$$f(x, y, c) = 0 \tag{1}$$

be the general solution of a differential equation of the first order,

$$\phi(x, y, p) = 0. \tag{2}$$

Then every curve of the family represented by (1) is such that the coördinates of every point of it and the slope of the curve at that point satisfy (2). If the family of curves has an envelope, the slope of the envelope at each point is that of a curve of the family. It follows that the envelope is a curve, such that the coördinates of every point of it and the slope of the curve at that point satisfy (2). *Hence the equation of the envelope is a solution of* (2). It is not a particular case of the general solution, since it cannot be obtained from the general solution by giving the constant a particular value, and is called the *singular solution.*

---

* Some writers call the whole locus of the limit points the envelope, while other writers define the envelope as a curve tangent to every curve of the family.

Accordingly, we may find the singular solution, if one exists, by finding the envelope of the family of curves represented by the general solution. This method requires us to find the general solution first; but we may find the singular solution, without knowing the general solution, as follows:

Let (1) and (2) (fig. 117) be two curves of the family represented by (1), intersecting at $P_1(x_1, y_1)$, and having the respective slopes p_1 and p_2. Then x_1, y_1, p_1, and x_1, y_1, p_2 satisfy (2). As curves (1) and (2) approach coincidence, in general, P_1 approaches a point of the envelope as a limit, and p_1 and p_2 become equal. Hence the locus of points for which (2), regarded as an equation in p, has equal roots must include the envelope, if one exists. The equation of this locus may be found by placing the discriminant of the equation, regarded as an equation in p, equal to zero.

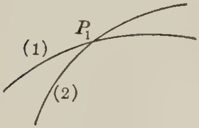

FIG. 117

As in the determination of envelopes, so here, extraneous factors may appear in the course of the work, and they can be eliminated most easily by testing them in the differential equation, to see if they satisfy it.

Ex. 1. Find the singular solution, if one exists, of the differential equation

$$y = px + a\sqrt{1 + p^2}.$$

First method. The general solution has been found to be (§ 177, Ex. 3)

$$y = cx + a\sqrt{1 + c^2};$$

and the envelope of this family of straight lines is (§ 179, Ex.) the circle $x^2 + y^2 - a^2 = 0$. Hence there is a singular solution, i.e.

$$x^2 + y^2 - a^2 = 0.$$

Second method. Writing the differential equation as a rational algebraic equation in p, we have

$$(x^2 - a^2)p^2 - 2xyp + (y^2 - a^2) = 0,$$

the discriminant of which is $4a^2(x^2 + y^2 - a^2)$.

Since $x^2 + y^2 - a^2 = 0$ satisfies the differential equation, it is the singular solution.

Ex. 2. In solving Clairaut's equation (§ 177, Ex. 2), we neglected the factor $x+f'(p)$. The equation $x+f'(p)=0$, however, is the equation which would be derived if Clairaut's equation were differentiated with respect to p. Hence the elimination of p between this and Clairaut's equation would give us an equation which would include the singular solution, if one exists. In Ex. 1 of § 177 we found the solution $2ax \pm 2y\sqrt{a}=0$. This is now seen to be a singular solution of the given equation.

183. Orthogonal trajectories. A curve which intersects each curve of the family represented by the equation

$$f(x, y, c)=0 \tag{1}$$

at a given angle is called a *trajectory*. In particular, if the given angle is a right angle, the curve is called an *orthogonal trajectory*. It is only this special class that we shall consider.

To determine the equation of the family of orthogonal trajectories, we first find the differential equation of the family represented by (1) in the form (§ 173)

$$F\left(x, y, \frac{dy}{dx}\right)=0. \tag{2}$$

Since the trajectory and the curve of the family intersect at right angles, the slope of the curve of the family is minus the reciprocal of the slope of the trajectory. Hence, if we replace $\frac{dy}{dx}$ in (2) by $-\frac{dx}{dy}$, the resulting equation

$$F\left(x, y, -\frac{dx}{dy}\right)=0 \tag{3}$$

is the differential equation of the family of orthogonal trajectories. The solution of (3) is the equation of the orthogonal trajectories.

Ex. Find the orthogonal trajectories of the family of hyperbolas $xy=a^2$.

The differential equation of this family of hyperbolas is

$$x\frac{dy}{dx}+y=0.$$

Hence the differential equation of the orthogonal trajectories is

$$x\left(-\frac{dx}{dy}\right)+y=0,$$

the solution of which is $$x^2-y^2=c.$$

Hence the orthogonal trajectories are hyperbolas, concentric with the given hyperbolas and having their common axis making an angle of 45° with the common axis of the given hyperbolas (fig. 113).

184. Differential equation of the first order in three variables. The integrable case. Any family of surfaces

$$f(x, y, z, c) = 0 \tag{1}$$

satisfies a differential equation of the form

$$Pdx + Qdy + Rdz = 0. \tag{2}$$

For, by § 114, $$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0,$$

and the elimination of c from this equation by means of (1) gives (2).

Conversely, we ask if an equation of the form (2) always has a solution of the form (1). To answer this question, we notice that (1) may be written $$\phi(x, y, z) = c,$$

whence $$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = 0, \tag{3}$$

which is an *exact differential equation* (§ 151).

As (3) does not contain c, it must either be the equation (2) or differ from it by some factor. Hence equation (2) has an integral (1) only when it is exact or can be made exact by means of a factor, called an *integrating factor*. A special case of an exact differential equation is one in which the variables are separated. We shall accordingly consider three cases of equation (2), namely:

Case I, equations in which the variables can be separated.
Case II, exact equations.
Case III, equations having integrating factors.

CASE I. If the variables can be separated so that the equation may be written in the form

$$f_1(x)\,dx + f_2(y)\,dy + f_3(z)\,dz = 0, \tag{4}$$

where any coefficient may reduce to a constant, the solution is evidently of the form

$$\int f_1(x)\,dx + \int f_2(y)\,dy + \int f_3(z)\,dz = c. \tag{5}$$

Ex. 1. $(x+a)yz\,dx+(x-a)(y+b)z\,dy+(x-a)(z+c)y\,dz=0.$

Dividing the equation by $(x-a)yz$, we have

$$\frac{x+a}{x-a}dx+\frac{y+b}{y}dy+\frac{z+c}{z}dz=0.$$

Hence the solution is

$$\int\frac{x+a}{x-a}dx+\int\frac{y+b}{y}dy+\int\frac{z+c}{z}dz=k,$$

or
$$x+2a\log(x-a)+y+b\log y+z+c\log z=k,$$

or
$$x+y+z+\log[(x-a)^{2a}y^bz^c]=k.$$

Case II. The necessary and sufficient conditions that (2) shall be an exact differential equation are (§ 151)

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x},\quad \frac{\partial Q}{\partial z}=\frac{\partial R}{\partial y},\quad \frac{\partial R}{\partial x}=\frac{\partial P}{\partial z}. \tag{6}$$

These conditions being fulfilled, equation (2) is of the form

$$\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz=0, \tag{7}$$

and the problem is to find ϕ.

If we omit from (2) one term, say Rdz, we have the equation

$$Pdx+Qdy=0, \tag{8}$$

which, because of (6), is an exact differential equation (§ 147) obtained from (7) by considering z as constant. Therefore, if we integrate (8), holding z constant, we shall have all that part of ϕ which contains either x or y. The arbitrary constant in the solution of (8) must be replaced by an arbitrary function of z, since "constant" in this connection means "independent of x and y." If, then, the solution of (8) is

$$\phi_1(x, y, z)=\phi_2(z),$$

where ϕ_1 is a known and ϕ_2 an unknown function, we have

$$\phi=\phi_1(x, y, z)-\phi_2(z).$$

Substituting in (7) we should have equation (2), and comparison with (2) will give an equation from which to determine $\phi_2(z)$.

Ex. 2. $(y^2+z^2)\,x\,dx+(z^2+x^2)\,y\,dy+(x^2+y^2)\,z\,dz=0.$

This equation is exact.

Omitting the last term, we have the exact equation

$$(y^2+z^2)\,x\,dx+(z^2+x^2)\,y\,dy=0,$$

the solution of which is $\quad x^2y^2+z^2x^2+y^2z^2+F(z)=0.$

Forming an exact equation from this solution, we have

$$2\,(y^2+z^2)\,x\,dx+2\,(z^2+x^2)\,y\,dy+[2\,(y^2+x^2)\,z+F'(z)]\,dz=0.$$

Comparing this equation with the given equation, we have $F'(z)=0$, whence $F(z)=c$. Therefore the general solution is

$$x^2y^2+z^2x^2+y^2z^2=k.$$

Case III. If equation (2) has an integrating factor μ, then

$$\mu P dx+\mu Q dy+\mu R dz=0 \tag{9}$$

is an exact differential equation, and therefore

$$\frac{\partial}{\partial y}(\mu P)=\frac{\partial}{\partial x}(\mu Q),\quad \frac{\partial}{\partial z}(\mu Q)=\frac{\partial}{\partial y}(\mu R),\quad \frac{\partial}{\partial x}(\mu R)=\frac{\partial}{\partial z}(\mu P). \tag{10}$$

Equations (10) may be placed in the form

$$\mu\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=Q\frac{\partial \mu}{\partial x}-P\frac{\partial \mu}{\partial y},$$

$$\mu\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)=R\frac{\partial \mu}{\partial y}-Q\frac{\partial \mu}{\partial z},$$

$$\mu\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)=P\frac{\partial \mu}{\partial z}-R\frac{\partial \mu}{\partial x}.$$

Multiplying the first equation by R, the second equation by P, and the third equation by Q, and adding the three resulting equations, we have

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0. \tag{11}$$

This is then a necessary condition that must be fulfilled in order that (9) may be an exact differential equation or that (2) may have an integrating factor. It may be shown that the condition (11) is also sufficient; that is, if (11) is fulfilled, equation (2) has an integrating factor.

Let us now suppose that (11) is satisfied for a given equation of form (2). Then if it were possible to find the integrating factor μ, we should form the equation (9), and, omitting the last term, should solve the exact differential equation

$$\mu P dx + \mu Q dy = 0,$$

as in Case II. But since μ is not known, we may solve the equivalent equation

$$P dx + Q dy = 0,$$

writing the solution in the form $f(x, y, F(z)) = 0$, where $F(z)$ takes the place of the arbitrary constant. From this point on, the work is similar to that in Case II.

Ex. 3. $yz^2dx + (y^2z - xz^2)\,dy - y^3dz = 0.$

This equation has an integrating factor.

Regarding y as constant, we have the equation

$$yz^2dx - y^3dz = 0,$$

the solution of which is

$$x + \frac{y^2}{z} + F(y) = 0.$$

From this solution we form the differential equation

$$dx + \left[\frac{2y}{z} + F'(y)\right]dy - \frac{y^2}{z^2}dz = 0.$$

If we divide the given equation by yz^2, we have

$$dx + \left[\frac{y}{z} - \frac{x}{y}\right]dy - \frac{y^2}{z^2}dz = 0,$$

whence, by comparison,

$$\frac{y}{z} + \frac{x}{y} + F'(y) = 0.$$

But

$$\frac{x}{y} + \frac{y}{z} + \frac{F(y)}{y} = 0,$$

so that

$$F'(y) - \frac{F(y)}{y} = 0,$$

whence

$$F(y) = cy.$$

Therefore the general solution of the given equation is

$$x + \frac{y^2}{z} + cy = 0,$$

or

$$\frac{x}{y} + \frac{y}{z} + c = 0.$$

The student should notice the difference between the equations

$$M dx + N dy = 0$$

and

$$P dx + Q dy + R dz = 0.$$

The former has always an integrating factor and a solution $f(x, y, c) = 0$ (§ 173). The latter has an integrating factor and a solution $f(x, y, z, c) = 0$ only when condition (11) is satisfied.

185. Two differential equations of the first order in three variables. Let the two equations be

$$\begin{aligned} P_1dx + Q_1dy + R_1dz &= 0, \\ P_2dx + Q_2dy + R_2dz &= 0, \end{aligned} \tag{1}$$

where $P_1, Q_1, R_1, P_2, Q_2, R_2$ are functions of x, y, and z, or constants.

By I, § 8, we have

$$dx : dy : dz = \begin{vmatrix} Q_1 & R_1 \\ Q_2 & R_2 \end{vmatrix} : \begin{vmatrix} R_1 & P_1 \\ R_2 & P_2 \end{vmatrix} : \begin{vmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{vmatrix},$$

or

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}, \tag{2}$$

where $P = Q_1R_2 - Q_2R_1$, $Q = R_1P_2 - R_2P_1$, and $R = P_1Q_2 - P_2Q_1$. Accordingly, we shall consider equations in form (2) only.

Since $dx : dy : dz$ gives a direction in space (§ 97), (2) assigns a specific direction at each point in space. Moving from point to point in the direction determined by (2), we trace a curve in space. Hence the solution of (2) consists of a family of space curves, and since it requires two simultaneous equations to represent such a curve (§ 89), it follows that the solution of (2) is a pair of simultaneous equations.

If the first of the equations (2), i.e. $\dfrac{dx}{P} = \dfrac{dy}{Q}$, is independent of z, it is an equation in two variables, the solution of which may be written in the form

$$f_1(x, y, c_1) = 0. \tag{3}$$

Similarly, if the remaining equations of (2) are independent of x or of y, their solutions are respectively

$$f_2(y, z, c_2) = 0, \tag{4}$$

and

$$f_3(x, z, c_3) = 0. \tag{5}$$

Any two of the three equations (3), (4), (5) taken simultaneously constitute the solution of (2).

Ex. 1. $\frac{dx}{xy}=\frac{dy}{y}=\frac{dz}{z}$.

From $\frac{dy}{y}=\frac{dz}{z}$ we have $y=c_1z$.

The equation $\frac{dx}{xy}=\frac{dy}{y}$ may be written $\frac{dx}{x}=dy$, whence

$$x=c_2e^y.$$

Therefore the complete solution consists of the equations

$$y=c_1z, \qquad x=c_2e^y$$

taken simultaneously.

If only one of the equations (2) can be solved by the above method, we may proceed as follows: Suppose, for example, that we have solved the first of equations (2) with the result (3); we may then solve (3) for either x or y and substitute in one of the two remaining equations, thus forming an equation in y, z, and c_1 or x, z, and c_1, which can be solved. This solution taken simultaneously with (3) constitutes the solution of (2).

Ex. 2. $\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{xyze^x}$.

The solution of $\frac{dx}{x}=\frac{dy}{y}$ is $y-c_1x=0$.

Equating the first and the third fractions, we have $dx=\frac{dz}{yze^x}$. Substituting c_1x for y in this equation, we have $c_1xe^xdx=\frac{dz}{z}$, whence

$$c_1(x-1)\,e^x=\log c_2z.$$

Therefore the complete solution consists of the equations

$$y-c_1x=0, \qquad c_1(x-1)\,e^x-\log c_2z=0$$

taken simultaneously.

If both of the previous methods fail, we may proceed as follows: By the theory of proportion we may write

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{k_1dx+k_2dy+k_3dz}{k_1P+k_2Q+k_3R},$$

where k_1, k_2, k_3 are arbitrary functions of x, y, and z, or constants. There are three cases to consider:

1. k_1, k_2, k_3 may be so chosen as to give between the fourth fraction and one of the original fractions an equation which can be solved (see Ex. 3).

2. k_1, k_2, k_3 may be so chosen as to make $k_1P + k_2Q + k_3R = 0$. Then $k_1dx + k_2dy + k_3dz = 0$, and if this equation falls under one of the cases of § 184, its solution is one of the equations of the solution of the given differential equations (see Ex. 4).

3. We may form a new equation

$$\frac{k_1dx + k_2dy + k_3dz}{k_1P + k_2Q + k_3R} = \frac{k_1'dx + k_2'dy + k_3'dz}{k_1'P + k_2'Q + k_3'R},$$

so choosing the multipliers k_1, k_2, k_3, k_1', k_2', k_3' as to make the new equation solvable by previous methods (see Ex. 5).

Ex. 3. $\frac{dx}{x} = \frac{dy}{x+y} = \frac{dz}{x+z}$.

Let $$k_1 = 0,\ k_2 = 1,\ k_3 = -1.$$

Then $$\frac{dx}{x} = \frac{dy}{x+y} = \frac{dz}{x+z} = \frac{dy - dz}{y - z} = \frac{d(y-z)}{y-z}.$$

From $\frac{dx}{x} = \frac{d(y-z)}{y-z}$, we find $\quad x = c_1(y - z)$.

Substituting this value of x in the equation $\frac{dy}{x+y} = \frac{dz}{x+z}$, we have

$$\frac{dy}{(1 + c_1)y - c_1z} = \frac{dz}{(1 - c_1)z + c_1y},$$

the solution of which is $\quad \log(z - y) = c_2 - \frac{y}{c_1(z-y)}$.

Therefore the general solution consists of the two equations

$$x = c_1(y - z), \qquad \log(z - y) = c_2 - \frac{y}{c_1(z-y)}$$

taken simultaneously.

Ex. 4. $\frac{dx}{y+z} = \frac{dy}{-x} = \frac{dz}{x+y+z}$.

Let $$k_1 = 1,\ k_2 = -1,\ k_3 = -1.$$

Then $$k_1(y + z) + k_2(-x) + k_3(x + y + z) = 0,$$

and hence $$k_1dx + k_2dy + k_3dz = 0.$$

But this equation is $$dx - dy - dz = 0,$$

the solution of which is evidently $\quad x - y - z = c_1$.

Substituting the value of $y + z$ from this equation in $\frac{dx}{y+z} = \frac{dy}{-x}$, we have $\frac{dx}{x - c_1} = \frac{dy}{-x}$, the solution of which is

$$x + c_1 \log(x - c_1) = c_2 - y.$$

Hence the general solution consists of the equations

$$x - y - z = c_1, \qquad x + c_1 \log(x - c_1) = c_2 - y.$$

Ex. 5. $\frac{dx}{y+z}=\frac{dy}{z+x}=\frac{dz}{x+y}$.

Let $$k_1=k_2=k_3=1.$$

Then $$\frac{dx}{y+z}=\frac{dy}{z+x}=\frac{dz}{x+y}=\frac{d(x+y+z)}{2(x+y+z)}.$$

Again, let $k_1=1$, $k_2=-1$, $k_3=0$, and we obtain the equal fraction $\frac{d(x-y)}{y-x}$; also, letting $k_1=0$, $k_2=1$, $k_3=-1$, we obtain the equal fraction $\frac{d(y-z)}{z-y}$.

$$\therefore \frac{d(x+y+z)}{2(x+y+z)}=\frac{d(x-y)}{y-x}=\frac{d(y-z)}{z-y},$$

whence $$\sqrt{x+y+z}=\frac{c_1}{x-y},$$

$$\sqrt{x+y+z}=\frac{c_2}{y-z}.$$

186. Differential equation of the first order in three variables. The nonintegrable case. Consider again the equation

$$Pdx+Qdy+Rdz=0. \tag{1}$$

Geometrically, the equation asserts that the direction $dx:dy:dz$ is perpendicular to the direction $P:Q:R$. Consequently, the geometrical solution of (1) consists of loci perpendicular to the curves defined by the equations

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}. \tag{2}$$

We may therefore seek the solution of (1) first in a family of surfaces

$$f(x, y, z, c)=0, \tag{3}$$

which will be orthogonal to the curves (2). This is the form of the solution discussed in § 184, and does not always exist. This leads to the geometric theorem that it is not always possible to find a family of surfaces orthogonal to a given family of curves.

When the solution of (1) in the form (3) does not exist, it is still possible to find curves which satisfy (1) and hence cut the curves (2) at right angles. In fact, we may find a family of such curves on any surface assumed at pleasure. For let

$$\phi(x, y, z)=0 \tag{4}$$

be the equation of any arbitrarily assumed surface. Then, from (4),

$$\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz=0. \tag{5}$$

Equations (1) and (5) may then be taken simultaneously. Their solution will be a family of curves which lie on (4) and satisfy condition (1).

Ex. $$xydx + ydy + zdz = 0. \tag{1}$$

This equation cannot be satisfied by a family of surfaces. It may, however, be satisfied by curves which lie on any assumed surface and cut at right angles the curves
$$\frac{dx}{xy} = \frac{dy}{y} = \frac{dz}{z}, \tag{2}$$
or (Ex. 1, § 185), $$y = c_1 z, \qquad x = c_2 e^y.$$
Let us assume the sphere $$x^2 + y^2 + z^2 = a^2. \tag{3}$$
Then $$xdx + ydy + zdz = 0, \tag{4}$$
and from (4) and (1), $$dx = 0, \text{ whence } x = c.$$

Hence the circles cut from the sphere $x^2 + y^2 + z^2 = a^2$ by the planes $x = c$ satisfy (1).

Again, let us assume the hyperbolic paraboloid
$$z = xy. \tag{5}$$
Then $$ydx + xdy - dz = 0, \tag{6}$$
and, from (6) and (1), $$\frac{dx}{-y - xz} = \frac{dy}{xy + zy} = \frac{dz}{x^2y - y^2}. \tag{7}$$

One solution of (6) is known to be $z = xy$. Using this, we have
$$\frac{xdx}{1 + x^2} = -\frac{dy}{1 + y},$$
whence $$(1 + y)\sqrt{1 + x^2} = c. \tag{8}$$
Then the curves defined by (5) and (8) also satisfy (1).

PROBLEMS

Express the solution of each of the following equations in the form of a series:

1. $\dfrac{dy}{dx} = y^2 - x.$ **2.** $y\dfrac{dy}{dx} = x^2 + y.$ **3.** $\dfrac{dy}{dx} = x^3 + y^2.$

Solve the following equations:

4. $p^2 - 3p - 10 = 0.$

5. $xy(p^2 + 1) - (x^2 + y^2)p = 0.$

6. $x^2p^2 + xyp - 2y^2 = 0.$

7. $p^3 + 2yp^2 - x^2p^2 - 2x^2yp = 0.$

8. $p^2(x^2 - a^2)^2 - 4a^2 = 0.$

9. $p^2 + 2py \operatorname{ctn} x - y^2 = 0.$

10. $py^2 - 2p^2xy + p^3x^2 = 1.$

11. $y(1 + p^2) - 2px = 0.$

12. $y = yp^2 + 2px.$

13. $(1 + y^2)p^2 - 2xyp^3 + x^2p^4 = 1.$

14. $2y - 2p = p^2.$

15. $p^3 - 4xyp + 8y^2 = 0.$

16. $x^2p^2 - p^2 + 1 = 0.$ **18.** $p^2 + 2\,px^3 = 4\,x^2y.$

17. $y\,(p + y) - p^2x = 0.$ **19.** $p\,(x + y)^2 = 1.$

20. $x = 4\,p^2 + 4\,p.$ **22.** $y = p^2x + p.$

21. $x^2yp^2 - x^2\,(x + y)\,p + (x - y)\,(x + y)^2 = 0.$ **23.** $y = p^2 - 2\,p.$

24. $p^3 - (x^2 + xy + y^2)\,p^2 + xy\,(x^2 + xy + y^2)\,p - x^3y^3 = 0.$

25. $y + 2\,py - 2\,p^2x = 0.$ **26.** $p^2 - 2\,xp + x^2 + 4\,y^2 = 0.$

27. $y^4 - 4\,xy^3p + 4\,p^2y^2\,(x^2 - 1) + 1 = 0.$ (Let $y^2 = z.$)

28. $e^{3\,x}\,(p - 1) + p^3e^{2\,y} = 0.$ (Let $e^y = v,\ e^x = u.$)

29. $py^2\,(px - y) + x = 0.$ (Let $y^2 = v,\ x^2 = u.$)

Find the general and the singular solutions of the following equations:

30. $p^2\,(1 + x^2) - 2\,xyp + y^2 - 1 = 0.$ **34.** $y^2 + (1 - 2\,xy)\,p + x^2p^2 = 0.$

31. $3\,x^2 - 4\,xp + p^2 = 0.$ **35.** $a^2 - y^2 = p^2y^2.$

32. $(x^2 - a^2)\,p^2 - 2\,xyp - x^2 = 0.$ **36.** $x^3p^2 + x^2yp + a^3 = 0.$

33. $y = -\,xp + x^4p^2.$

37. Find the singular solution of the equation $p^3 - 4\,xyp + 8\,y^2 = 0.$

38. Find the singular and the general solutions of the equation $y = px + \sqrt{b^2 + a^2p^2}$, and interpret them geometrically.

Solve the following equations:

39. $(xy^2z - xz)\,dx + (x^2yz + yz)\,dy + (x^2y^2 - x^2 + y^2 - 1)\,dz = 0.$

40. $(y + z)(2\,x + y + z)\,dx + (z + x)(2\,y + z + x)\,dy + (x + y)(2\,z + x + y)\,dz = 0.$

41. $3\,x^2yz\,dx + (2\,y^2z + z)\,dy + 2\,(y + yz^2)\,dz = 0.$

42. $(y + z - b - c)\,dx + (z + x - c - a)\,dy + (x + y - a - b)\,dz = 0.$

43. $(y^2 + yz)\,dx + (z^2 + zx)\,dy + (y^2 - xy)\,dz = 0.$

44. $\left(\frac{1}{y} - \frac{z}{x^2}\right)dx + \left(\frac{1}{z} - \frac{x}{y^2}\right)dy + \left(\frac{1}{x} - \frac{y}{z^2}\right)dz = 0.$

45. $(1 + x + y)\,dx + (1 + x + y)\,dy + (x + y)\,dz = 0.$

46. $yz^2dx + (y^2z - xz^2)\,dy - y^2\,(y + z)\,dz = 0.$

47. $yz\,dx - zx\,dy + (x^2 + y^2)\,dz = 0.$

Solve the following systems of simultaneous equations:

48. $\dfrac{dx}{y\,(x^2+a^2)} = \dfrac{dy}{x\,(y^2+b^2)} = \dfrac{dz}{x(z^2+c^2)}.$ **50.** $\dfrac{dx}{y} = \dfrac{dy}{x + z} = \dfrac{dz}{y}.$

49. $\dfrac{dx}{yz} = \dfrac{dy}{zx} = \dfrac{dz}{xy}.$ **51.** $\dfrac{z^2dx}{\frac{1}{e^x}} = \dfrac{2\,x^2yz^2dy}{e^{y^2}} = \dfrac{x^2dz}{\frac{1}{e^z}}.$

52. $\dfrac{dx}{x^2 + 1} = \dfrac{dy}{xy + (x^2 + 1)^{\frac{3}{2}}} = \dfrac{dz}{y + xz - x\sqrt{x^2 + 1}}.$

53. $\frac{dx}{z}=\frac{xdy}{x^2+z^2}=-\frac{dz}{x}.$

54. $\frac{dx}{x}=\frac{dy}{2x-y}=\frac{dz}{z-xy}.$

55. $\frac{dx}{x+y-z}=\frac{dy}{z}=dz.$

56. $\frac{dx}{x-2y}=-\frac{dy}{y}=zdz.$

57. $\frac{dx}{x^2+y^2}=\frac{dy}{2xy}=\frac{dz}{(x-y)z}.$

58. $\frac{dx}{y-z}=\frac{dy}{z-x}=\frac{dz}{x-y}.$

59. $\frac{dx}{x-y-z}=\frac{dy}{y-z-x}=\frac{dz}{z}.$

60. $\frac{dx}{x^2-y^2-z^2}=\frac{dy}{2xy}=\frac{dz}{2xz}.$

61. Find the envelope of the family of lines $y=2mx+m^4$, m being the variable parameter.

62. Prove that each line of the family $c^2x+a^2y-ac^2=0$, a being the variable parameter, forms with the coördinate axes a triangle of constant area, and find the envelope of the family.

63. Find the envelope of the parabolas $y^2=a(x-a)$.

64. A straight line moves so that the sum of its intercepts on the coördinate axes is always equal to the constant c. Express the equation of the family of lines in terms of c and the intercept on OX, and find their envelope.

65. The semiaxes of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ vary so that $ab=c^2$, where c is a constant. Express the equation of the ellipse in terms of a as a variable parameter, and find the envelope of the family of ellipses thus defined.

66. Find the envelope of the family of straight lines formed by varying the slope m in the equation $y=mx-2pm-pm^3$.

67. Find the envelope of a family of circles having their centers on the line $y=2x$ and tangent to the axis of y.

68. Find the envelope of a family of circles which have the double ordinates of the parabola $y^2=4px$ as diameters.

69. Find the envelope of a family of straight lines which move so that the portion of each of them included between the axes is always equal to the constant c.

70. Find the envelope of a family of circles which have their centers on the parabola $y^2=4px$ and pass through the vertex of the parabola.

71. Find the equation of a curve such that the tangent cuts off from the coördinate axes intercepts the sum of which is always equal to the constant k.

72. Find a curve in which the projection upon OY of the perpendicular from the origin upon any tangent is always equal to the constant a.

73. Find the equation of the curve in which the part of the tangent included between the coördinate axes is always equal to the constant a.

74. Determine the equation of a curve such that the portion of the axis of x intercepted by the tangent and the normal at any point of the curve is always equal to the constant k.

75. Find the polar equation of the curve in which the perpendicular from the pole upon any tangent is always equal to the constant k.

76. Find the polar equation of a curve such that the perpendicular from the pole upon any tangent is k times the radius vector of the point of contact.

77. Find the orthogonal trajectories of the family of parabolas $y^2 = 4px$.

78. Find the orthogonal trajectories of the family of ellipses $\frac{x^2}{a^2} + \frac{y^2}{a^2 + \lambda} = 1$, λ being the variable parameter.

79. Find the orthogonal trajectories of the family of ellipses $\frac{x^2}{a^2} + \frac{4y^2}{a^2} = 1$.

80. Find the orthogonal trajectories of the family of parabolas $y^2 = 4ax + 4a^2$.

81. Find the orthogonal trajectories of a family of circles each of which is tangent to the axis of y at the origin.

82. Find the orthogonal trajectories of the family of circles each of which passes through the points $(\pm 1, 0)$.

83. If $f\left(r, \theta, \frac{dr}{d\theta}\right) = 0$ is the equation of a family of curves, prove that $f\left(r, \theta, -r^2\frac{d\theta}{dr}\right) = 0$ is the equation of the orthogonal trajectories.

84. Find the orthogonal trajectories of the family of lemniscates $r^2 = 2a^2 \cos 2\theta$.

85. Find the orthogonal trajectories of the family of cardioids $r = a(\cos\theta + 1)$.

86. Find the orthogonal trajectories of the family of logarithmic spirals $r = e^{a\theta}$.

CHAPTER XVIII

THE LINEAR DIFFERENTIAL EQUATION

187. Definitions. The equation

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_{n-1} \frac{dy}{dx} + p_n y = f(x), \tag{1}$$

where $p_1, p_2, \cdots, p_{n-1}, p_n$, and $f(x)$ are independent of y, is a *linear differential equation.* If the coefficients $p_1, p_2, \cdots, p_{n-1}, p_n$ are constants, the equation becomes the *linear differential equation with constant coefficients,*

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = f(x), \tag{2}$$

where $a_1, a_2, \cdots, a_{n-1}, a_n$ are constants.

In both (1) and (2) $f(x)$ is a function of x, which may reduce to a constant or even be zero.

We shall begin with the study of (2). To do this, it is convenient to express $\frac{dy}{dx}$ by Dy, $\frac{d^2y}{dx^2}$ by D^2y, $\cdots$, $\frac{d^ny}{dx^n}$ by D^ny, and to rewrite (2) in the form

$$D^n y + a_1 D^{n-1} y + \cdots + a_{n-1} D y + a_n y = f(x),$$

or, more compactly,

$$(D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n) y = f(x). \tag{3}$$

The expression in parenthesis in (3) is called an *operator*, and we are said to operate upon a quantity with it when we carry out the indicated operations of differentiation, multiplication, and addition. Thus, if we operate on $\sin x$ with $D^3 - 2D^2 + 3D - 5$, we have

$$\begin{aligned}(D^3 - 2D^2 + 3D - 5)\sin x &= -\cos x + 2\sin x + 3\cos x - 5\sin x \\ &= 2\cos x - 3\sin x.\end{aligned}$$

Also, the solution of (2) or (3) is expressed by the equation

$$y=\frac{1}{D^n+a_1D^{n-1}+\cdots+a_{n-1}D+a_n}f(x), \tag{4}$$

where the expression on the right hand of this equation is not to be considered as a fraction but simply as a symbol to express the solution of (3). Thus if (3) is the very simple equation $Dy=f(x)$, then (4) becomes

$$y=\frac{1}{D}f(x)=\int f(x)\,dx.$$

In this case $\frac{1}{D}$ means integration with respect to x. What the more complicated symbol (4) may mean, we are now to study.

188. The equation of the first order with constant coefficients. The linear equation of the first order with constant coefficients is

$$\frac{dy}{dx}-ay=f(x),$$

or, symbolically,

$$(D-a)y=f(x). \tag{1}$$

This is a special case of the linear equation discussed in § 80, and we have only to place $f_1(x)=-a$, $f_2(x)=f(x)$ in formula (5) of § 80 to obtain the solution. We have in this way

$$y=\frac{1}{D-a}f(x)=ce^{ax}+e^{ax}\int e^{-ax}f(x)\,dx. \tag{2}$$

The solution (2) consists of two parts. The first part, ce^{ax}, contains an arbitrary constant, does not contain $f(x)$, and, if taken alone, is not a solution of (1) unless $f(x)$ is zero. The second part, $e^{ax}\int e^{-ax}f(x)\,dx$, contains $f(x)$, and, taken alone, is a solution of (1), since (1) is satisfied by (2) when c has any value including 0. Hence $e^{ax}\int e^{-ax}f(x)\,dx$ is called a *particular integral* of (1), and, in distinction to this, ce^{ax} is called the *complementary function.* The sum of the complementary function and the particular integral is the *general solution* (2). The complementary function can be written down from the left-hand member of equation (1), but the determination of the particular integral requires integration.

Ex. 1. Solve $\dfrac{dy}{dx} + 3y = 5x^3$.

The complementary function is ce^{-3x}. The particular integral is

$$5e^{-3x}\int e^{3x}x^3dx = \tfrac{5}{3}x^3 - \tfrac{5}{3}x^2 + \tfrac{10}{9}x - \tfrac{10}{27}.$$

Hence the general solution is $y = ce^{-3x} + \frac{5}{3}x^3 - \frac{5}{3}x^2 + \frac{10}{9}x - \frac{10}{27}$.

Ex. 2. Solve $\dfrac{dy}{dx} + y = \sin x$.

The complementary function is ce^{-x}. The particular integral is

$$e^{-x}\int e^x \sin x\, dx = \tfrac{1}{2}\sin x - \tfrac{1}{2}\cos x. \qquad (\S\ 19, \text{Ex. } 5)$$

Therefore the general solution is $y = ce^{-x} + \frac{1}{2}\sin x - \frac{1}{2}\cos x$.

189. The operator $\dfrac{1}{D-a}$. The solution of linear equations of higher order with constant coefficients depends upon the solution of the equation of the first order. Hence a knowledge of the operator $\dfrac{1}{D-a}$ is of prime importance. We give a few of the results obtained by operating with $\dfrac{1}{D-a}$ upon certain elementary functions which occur frequently in practice. In writing these formulas the complementary function, which is ce^{ax} in all cases, is omitted.

$$\frac{1}{D-a}cu = c\frac{1}{D-a}u. \tag{1}$$

$$\frac{1}{D-a}(u+v+w+\cdots) = \frac{1}{D-a}u + \frac{1}{D-a}v + \frac{1}{D-a}w + \cdots. \tag{2}$$

$$\frac{1}{D-a}x^m = -\left(\frac{x^m}{a} + \frac{mx^{m-1}}{a^2} + \frac{m(m-1)x^{m-2}}{a^3} + \cdots\right), \text{ unless } a = 0. \tag{3}$$

$$\frac{1}{D}x^m = \frac{x^{m+1}}{m+1}. \tag{4}$$

$$\frac{1}{D-a}e^{kx} = \frac{e^{kx}}{k-a}, \text{ unless } k = a. \tag{5}$$

$$\frac{1}{D-a}e^{ax} = xe^{ax}. \tag{6}$$

$$\frac{1}{D-a}x^me^{kx} = e^{kx}\left(\frac{x^m}{k-a} - \frac{mx^{m-1}}{(k-a)^2} + \frac{m(m-1)x^{m-2}}{(k-a)^3} - \cdots\right), \tag{7}$$

unless $k = a$.

$$\frac{1}{D-a}x^me^{ax} = e^{ax}\frac{x^{m+1}}{m+1}. \tag{8}$$

$$\frac{1}{D-a}\sin kx = \frac{-a\sin kx - k\cos kx}{a^2+k^2}, \text{ unless } a = \pm ki. \tag{9}$$

$$\frac{1}{D-ki}\sin kx = \frac{xe^{kix}}{2i} - \frac{e^{-kix}}{4k}. \tag{10}$$

$$\frac{1}{D-a}\cos kx = \frac{-a\cos kx + k\sin kx}{a^2+k^2}, \text{ unless } a = \pm ki. \tag{11}$$

$$\frac{1}{D-ki}\cos kx = \frac{xe^{kix}}{2} - \frac{e^{-kix}}{4ki}. \tag{12}$$

These formulas may all be proved by substituting the special functions concerned in the general formula (§ 188, (2)). For (3), (7), (9), and (11) the student may refer to Exs. 4, 5, and 6, § 19. The derivation of (10) is as follows: By § 188, (2),

$$\frac{1}{D-ki}\sin kx = e^{kix}\int e^{-kix}\sin kx\,dx.$$

If we attempt to use the method of Ex. 5, § 19, it fails to work, but by replacing sin kx by its value in terms of the exponential functions (§ 169, (5)) we have

$$\frac{1}{D-ki}\sin kx = \frac{e^{kix}}{2i}\int(1-e^{-2kix})\,dx = \frac{xe^{kix}}{2i} - \frac{e^{-kix}}{4k}.$$

Formula (12) is derived in a similar manner.

190. The equation of the second order with constant coefficients. The symbol $(D-a)(D-b)y$ means that y is to be operated on with $D-b$ and the result operated on with $D-a$. Now $(D-b)y = \frac{dy}{dx} - by$, and hence

$$\begin{aligned}(D-a)(D-b)y &= \frac{d}{dx}\left(\frac{dy}{dx} - by\right) - a\left(\frac{dy}{dx} - by\right)\\ &= \frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby\\ &= [D^2 - (a+b)D + ab]y\\ &= (D^2 + pD + q)y,\end{aligned} \tag{1}$$

where $p = -(a+b)$, $q = ab$.

This result, obtained by considering the real meaning of the operators, is the same as if the operators $D-a$ and $D-b$ had been multiplied together, regarding D as an algebraic quantity. Similarly, we find

$$(D-b)(D-a)y = [D^2-(a+b)D+ab]y = (D-a)(D-b)y.$$

That is, the order in which the two operators $D-a$ and $D-b$ are used does not affect the result.

Moreover, if $(D^2+pD+q)y$ is given, it is possible to find a and b so that (1) is satisfied. In fact, we have simply to factor D^2+pD+q, considering D as an algebraic quantity.

This gives a method of solving the linear equation of the second order with constant coefficients. For such an equation has the form

$$\frac{d^2y}{dx^2}+p\frac{dy}{dx}+qy=f(x),$$

or, what is the same thing,

$$(D^2+pD+q)y=f(x), \tag{2}$$

where p and q are constants and $f(x)$ is a function of x which may reduce to a constant or be zero.

Equation (2) may be written

$$(D-a)(D-b)y=f(x),$$

whence, by § 188, (2),

$$(D-b)y=\frac{1}{D-a}f(x)=c'e^{ax}+e^{ax}\int e^{-ax}f(x)\,dx.$$

Again applying § 188, (2), we have

$$\begin{aligned} y &= \frac{1}{D-b}\left(c'e^{ax}+e^{ax}\int e^{-ax}f(x)\,dx\right) \\ &= c_2e^{bx}+e^{bx}\int e^{-bx}\left(c'e^{ax}+e^{ax}\int e^{-ax}f(x)\,dx\right)dx. \end{aligned} \tag{3}$$

There are now two cases to be distinguished:

I. If $a \neq b$, (3) becomes

$$y=c_2e^{bx}+c_1e^{ax}+e^{bx}\int\left(e^{(a-b)x}\int e^{-ax}f(x)\,dx\right)dx. \tag{4}$$

II. If $a=b$, (3) becomes

$$y=(c_2+c_1x)e^{ax}+e^{ax}\iint e^{-ax}f(x)\,dx^2. \tag{5}$$

In each case the solution consists of two parts. The one is the *complementary function* $c_1e^{ax}+c_2e^{bx}$ or $(c_2+c_1x)e^{ax}$, involving two arbitrary constants but not involving $f(x)$. It can be written down from the left-hand member of the equation, and is, in fact, the solution of the equation $(D-a)(D-b)y=0$. The other part of the general solution is the *particular integral*, and involves $f(x)$. Its computation by (4) or (5) necessitates two integrations.

Formula (4) holds whether a and b are real or complex. But when a and b are conjugate complex, it is convenient to modify the complementary function as follows: Let us place

$$a = m + in, \qquad b = m - in.$$

Then the complementary function is

$$\begin{aligned} c_1e^{(m+in)x} &+ c_2e^{(m-in)x} \\ &= e^{mx}(c_1e^{inx} + c_2e^{-inx}) \\ &= e^{mx}[c_1(\cos nx + i\sin nx) + c_2(\cos nx - i\sin nx)] \\ &= e^{mx}(C_1\cos nx + C_2\sin nx), \end{aligned} \tag{6}$$

where $C_1 = c_1 + c_2$, $C_2 = i(c_1 - c_2)$. Since c_1 and c_2 are arbitrary constants, so also are C_1 and C_2, and we obtain all real forms of the complementary function by giving real values to C_1 and C_2.

The form (6) may also be modified as follows: Whatever be the values of C_1 and C_2 we may always find an angle α such that $\cos\alpha = \dfrac{C_1}{\sqrt{C_1^2+C_2^2}}$, $\sin\alpha = \dfrac{C_2}{\sqrt{C_1^2+C_2^2}}$. Then (6) becomes

$$ke^{mx}\cos(nx-\alpha), \tag{7}$$

where α and $k = \sqrt{C_1^2+C_2^2}$ are new arbitrary constants. Or, we may find an angle β, such that $\sin\beta = \dfrac{-C_1}{\sqrt{C_1^2+C_2^2}}$, $\cos\beta = \dfrac{C_2}{\sqrt{C_1^2+C_2^2}}$. Then (6) becomes

$$ke^{mx}\sin(nx-\beta). \tag{8}$$

Ex. 1. $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y = e^x.$

This equation may be written

$$(D+2)(D+3)y = e^x.$$

The complementary function is therefore $c_1e^{-2x} + c_2e^{-3x}$. To find the particular integral, we proceed as follows:

$$(D+3)y = \frac{1}{D+2}e^x = e^{-2x}\int e^{3x}dx = \frac{1}{3}e^x.$$

$$y = \frac{1}{D+3}\left(\frac{1}{3}e^x\right) = e^{-3x}\int\frac{1}{3}e^{4x}dx = \frac{1}{12}e^x.$$

Therefore the general solution is

$$y = c_1e^{-2x} + c_2e^{-3x} + \tfrac{1}{12}e^x.$$

Ex. 2. $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=x.$

This equation may be written $(D+1)^2y=x$.

Therefore the complementary function is $(c_1+c_2x)e^{-x}$. To find the particular integral, we proceed as follows:

$$(D+1)y=\frac{1}{D+1}x=e^{-x}\int xe^x dx=x-1.$$

$$y=\frac{1}{D+1}(x-1)=e^{-x}\int(x-1)e^x dx=x-2.$$

Therefore the general solution is

$$y=(c_1+c_2x)e^{-x}+x-2.$$

Ex. 3. Consider the motion of a particle of unit mass acted on by an attracting force directed toward a center and proportional to the distance of the particle from the center, the motion being resisted by a force proportional to the velocity of the particle.

If we take s as the distance of the particle from the center of force, the attracting force is $-ks$ and the resisting force is $-h\frac{ds}{dt}$, where k and h are positive constants. Hence the equation of motion is

$$\frac{d^2s}{dt^2}=-ks-h\frac{ds}{dt},$$

or

$$(D^2+hD+k)s=0. \tag{1}$$

The factors of the operator in (1) are $\left(D+\frac{h}{2}-\frac{\sqrt{h^2-4k}}{2}\right)\left(D+\frac{h}{2}+\frac{\sqrt{h^2-4k}}{2}\right)$. We have therefore to consider three cases:

I. $h^2-4k<0$. The solution of (1) is then

$$s=e^{-\frac{ht}{2}}\left(C_1\cos\frac{\sqrt{4k-h^2}}{2}t+C_2\sin\frac{\sqrt{4k-h^2}}{2}t\right),$$

or

$$s=ae^{-\frac{ht}{2}}\sin\left(\frac{\sqrt{4k-h^2}}{2}t-\beta\right).$$

The graph of s has the general shape of that shown in I, § 155, fig. 161. The particle makes an infinite number of oscillations with decreasing amplitudes, which approach zero as a limit as t becomes infinite.

II. $h^2-4k>0$. The solution of (1) is then

$$s=c_1e^{-\left(\frac{h}{2}-\frac{\sqrt{h^2-4k}}{2}\right)t}+c_2e^{-\left(\frac{h}{2}+\frac{\sqrt{h^2-4k}}{2}\right)t}.$$

The particle makes no oscillations, but approaches rest as t becomes infinite.

III. $h^2-4k=0$. The solution of (1) is

$$s=(c_1+c_2t)e^{-\frac{h}{2}t}.$$

The particle approaches rest as t becomes infinite.

191. Solution by partial fractions. Another method of solving the equation

$$(D^2 + pD + q)y = f(x), \tag{1}$$

when the factors of the operator are unequal, is as follows:

We may express the solution in the form

$$y = \frac{1}{D^2 + pD + q} f(x) = \frac{1}{(D-a)(D-b)} f(x).$$

Now we have seen that in many ways the operator D may be handled as if it were an algebraic quantity. This raises the question whether it is proper to separate $\frac{1}{(D-a)(D-b)}$ into partial fractions. Algebraically we have, of course,

$$\frac{1}{(D-a)(D-b)} = \frac{1}{a-b}\left(\frac{1}{D-a} - \frac{1}{D-b}\right),$$

and the question is, Is

$$y = \frac{1}{a-b}\left(\frac{1}{D-a} - \frac{1}{D-b}\right) f(x) \tag{2}$$

a solution of equation (1)?

The way to answer this question is to substitute this value of y in (1) and observe the result. We have then, on the left-hand side of (1),

$$\begin{aligned}
&(D-a)(D-b)\left[\frac{1}{a-b}\,\frac{1}{D-a} f(x) - \frac{1}{a-b}\,\frac{1}{D-b} f(x)\right] \\
&= \frac{1}{a-b}\left[(D-b)(D-a)\frac{1}{D-a} f(x) - (D-a)(D-b)\frac{1}{D-b} f(x)\right] \\
&= \frac{1}{a-b}\left[(D-b) f(x) - (D-a) f(x)\right] \\
&= \frac{1}{a-b}\left[-bf(x) + af(x)\right] \\
&= f(x).
\end{aligned}$$

Consequently (2) is a solution of (1).

Writing (2) out in full, we have

$$y = c_1 e^{ax} + c_2 e^{bx} + \frac{1}{a-b} e^{ax} \int e^{-ax} f(x)\,dx - \frac{1}{a-b} e^{bx} \int e^{-bx} f(x)\,dx. \tag{3}$$

It is to be noted that the complementary function is the same as in § 190, (4), but the particular integral appears in another form.

This method fails if $a = b$.

Ex. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 6y = e^{2x}.$

Since this equation may be written

$$(D-2)(D+3)y = e^{2x},$$

the complementary function is $c_1e^{2x} + c_2e^{-3x}$. To find the particular integral, we proceed as follows:

$$\begin{aligned} y &= \frac{1}{D^2+D-6}e^{2x} \\ &= \frac{1}{5}\left[\frac{1}{D-2} - \frac{1}{D+3}\right]e^{2x} \\ &= \tfrac{1}{5}e^{2x}\int dx - \tfrac{1}{5}e^{-3x}\int e^{5x}dx \\ &= \frac{x}{5}e^{2x} - \frac{1}{25}e^{2x}. \end{aligned}$$

Therefore the general solution is

$$y = c_1e^{2x} + c_2e^{-3x} + \frac{x}{5}e^{2x} - \frac{1}{25}e^{2x}.$$

192. The general equation with constant coefficients. The methods of solving a linear equation of the second order with constant coefficients are readily extended to an equation of the nth order with constant coefficients. Such an equation is

$$\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_ny = f(x), \tag{1}$$

or, symbolically written,

$$(D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y = f(x). \tag{2}$$

The first step is to separate the operator in (2) into its linear factors and to write (2) as

$$(D-r_1)(D-r_2)\cdots(D-r_n) = f(x), \tag{3}$$

where $r_1, r_2, \cdots, r_n$ are the roots of the algebraic equation

$$r^n + a_1r^{n-1} + \cdots + a_{n-1}r + a_n = 0.$$

It may be shown, as in § 190, that the left-hand members of (2) and (3) are equivalent, and that the order of the factors in (3) is immaterial.

The general solution of (1) consists now of two parts, the complementary function and the particular integral.

The *complementary function* is written down from the factored form of the left-hand side of (3), and is the solution of (1) in the special case in which $f(x)$ is zero. If $r_1, r_2, \cdots, r_n$ are all distinct, the complementary function consists of the n terms

$$c_1e^{r_1x} + c_2e^{r_2x} + \cdots + c_ne^{r_nx}, \tag{4}$$

where $c_1, c_2, \cdots, c_n$ are arbitrary constants.

If, however, $D - r_i$ appears as a k-fold factor in (3), k of the terms of (4) must be replaced by the terms

$$(c_1 + c_2x + c_3x^2 + \cdots + c_kx^{k-1})\,e^{r_ix}.$$

Also, if two factors of (3) are conjugate complex numbers, the corresponding terms of (4) may be replaced by terms involving sines and cosines, as in (6), § 190.

The *particular integral* is found by evaluating

$$\frac{1}{(D - r_1)(D - r_2)\cdots(D - r_n)}f(x). \tag{5}$$

This may be done in two ways:

1. The expression (5) may be evaluated by applying the operators $\dfrac{1}{D - r_n}, \dfrac{1}{D - r_{n-1}}, \cdots$ in succession from right to left. This leads to a multiple integral of the form

$$e^{r_1x}\int e^{(r_2 - r_1)x}\int e^{(r_3 - r_2)x}\cdots\int e^{-r_nx}f(x)\,dx^n. \tag{6}$$

2. The operator in (5) may be separated into partial fractions. When the factors of (5) are all distinct, this leads to an integral of the type

$$A_1e^{r_1x}\int e^{-r_1x}f(x)\,dx + A_2e^{r_2x}\int e^{-r_2x}f(x)\,dx + \cdots$$
$$+ A_ne^{r_nx}\int e^{-r_nx}f(x)\,dx. \tag{7}$$

If some of the factors of (3) are repeated, the previous method must be combined with this.

In evaluating (6) and (7) the constants of integration may be omitted, since they are taken care of in the complementary function.

The *general solution* is the sum of the complementary function and the particular integral.

193. Solution by undetermined coefficients. The work of finding the particular integral may be much simplified when the form of the integral can be anticipated. The particular integral may then be written with unknown, or *undetermined,* coefficients, and the coefficients determined by direct substitution in the differential equation. Since both (6) and (7), § 192, consist of successive applications of the operator $\frac{1}{D-a}$, we may apply the formulas of § 189 in many cases. From (2), § 189, it follows that if $f(x)$ consists of an aggregate of terms, the particular integral is the sum of the parts obtained by taking each term by itself. From (1), § 189, it follows that the coefficient of a term of $f(x)$ affects only the coefficient of the corresponding part of the particular integral. From the other formulas of § 189 we can deduce the following:

1. When $f(x) = a_0x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m$ (m a positive integer), the particular integral is of the form $A_0x^m + A_1x^{m-1} + \cdots + A_{m-1}x + A_m$, unless the left-hand member of the differential equation contains the factor D^r. In the latter case the particular solution is of the form $x^r(A_0x^m + A_1x^{m-1} + \cdots + A_{m-1}x + A_m)$, while terms of the form $A_{m+1}x^{r-1} + \cdots + A_{m+r-1}x + A_{m+r}$ occur in the complementary function, and hence need not be assumed as part of the particular integral.

2. When $f(x) = ae^{kx}$, the particular integral is of the form Ae^{kx}, unless the left-hand member of the differential equation contains a factor $(D-k)^r$. In the latter case the particular integral is of the form Ax^re^{kx}, while terms of the form $(A_1x^{r-1} + \cdots + A_{r-1}x + A_r)e^{kx}$ occur in the complementary function, and hence need not be assumed in the particular integral.

3. When $f(x) = a \sin kx$ or $a \cos kx$, the particular integral is of the form $A \sin kx + B \cos kx$, unless the left-hand member of the differential equation contains the factor $(D^2 + k^2)^r$. In the latter

case the particular integral is of the form $x^r(A\sin kx+B\cos kx)$, while terms of the form $x^{r-1}(A_1\sin kx+B_1\cos kx)+\cdots+(A_r\sin kx+B_r\cos kx)$ occur in the complementary function, and hence need not be assumed in the particular integral.

4. When $f(x)=ax^me^{kx}$ (m a positive integer), the particular integral is of the form $(A_0x^m+A_1x^{m-1}+\cdots+A_{m-1}x+A_m)e^{kx}$, unless the left-hand member of the differential equation contains a factor $(D-k)^r$. In the latter case the particular integral is of the form $x^r(A_0x^m+A_1x^{m-1}+\cdots+A_{m-1}x+A_m)e^{kx}$, while terms of the form $(A_{m+1}x^{r-1}+\cdots+A_{m+r-1}x+A_{m+r})e^{kx}$ occur in the complementary function, and hence need not be assumed in the particular integral.

The above statements may all be summed up in the following rule, which may also be sometimes used when $f(x)$ is not one of the forms mentioned above:

If u is a term of $f(x)$, and if $u_1, u_2, \cdots, u_k$ are all the distinct functions (disregarding constant coefficients) which can be obtained from u by successive differentiation, then the corresponding part of the particular integral is of the form $Au+A_1u_1+\cdots+A_ku_k$, unless u is a term of the complementary function, or such a term multiplied by an integral power of x. In the latter case the corresponding part of the particular integral is of the form $x^r(Au+A_1u_1+\cdots+A_ku_k)$, where r is the number of times the factor, which gives in the complementary function the term u, or u divided by an integral power of x, appears in the left-hand member of the differential equation.

The above rule is of course valueless unless the functions $u_1, u_2, \cdots, u_k$ are finite in number. In applying it to functions other than those already discussed, the student should consider that he is making an experiment. If a function assumed in accordance with the rule is found to satisfy the differential equation, the use of the rule is justified. When it fails, recourse may always be had to the general formulas (6) and (7), § 192.

When $f(x)=e^{kx}\phi(x)$, the work of finding the particular integral may be lightened by substituting $y=e^{kx}z$ in the differential equation. Then, since all derivatives of $e^{kx}z$ contain the factor e^{kx}, the new differential equation may be divided by e^{kx}, and there is left an equation for z in which the left-hand member is $\phi(x)$. When this equation is solved for z, y is readily found (see Ex. 2).

Ex. 1. $\dfrac{d^3y}{dx^3} - 2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = xe^{2x} + e^x.$

This may be written $\quad D(D-1)^2y = xe^{2x} + e^x,$

whence the complementary function is $c_1 + (c_2 + c_3x)e^x$. The term xe^{2x}, if successively differentiated, gives only the new form e^{2x}. Hence the corresponding part of the particular integral has the form $Axe^{2x} + Be^{2x}$. The term e^x gives no new form if differentiated; but since it appears in the complementary function corresponding to the double factor $(D-1)^2$, the corresponding part of the particular integral is Cx^2e^x. Substituting

$$y = Axe^{2x} + Be^{2x} + Cx^2e^x$$

in the differential equation, we have

$$2Axe^{2x} + (5A + 2B)e^{2x} + 2Ce^x = xe^{2x} + e^x.$$

Then $\qquad 2A = 1, \qquad 5A + 2B = 0, \qquad 2C = 1,$

whence $\qquad A = \frac{1}{2}, \qquad B = -\frac{5}{4}, \qquad C = \frac{1}{2}.$

The general solution of the differential equation is, accordingly,

$$y = c_1 + (c_2 + c_3x)e^x + \tfrac{1}{2}xe^{2x} - \tfrac{5}{4}e^{2x} + \tfrac{1}{2}x^2e^x.$$

Ex. 2. $\dfrac{d^2y}{dx^2} + y = xe^{3x}\sin x.$

We place $y = e^{3x}z$. There results

$$\frac{d^2z}{dx^2} + 6\frac{dz}{dx} + 10z = x\sin x.$$

The complementary function is $e^{-3x}(c_1\sin x + c_2\cos x)$. For the particular integral we assume

$$z = Ax\sin x + Bx\cos x + C\sin x + E\cos x,$$

and, on substitution, have

$$(9A - 6B)x\sin x + (6A + 9B)x\cos x + (6A - 2B + 9C - 6E)\sin x$$
$$+ (2A + 6B + 6C + 9E)\cos x = x\sin x.$$

Therefore

$$\begin{aligned} 9A - 6B &= 1, \\ 6A + 9B &= 0, \\ 6A - 2B + 9C - 6E &= 0, \\ 2A + 6B + 6C + 9E &= 0, \end{aligned}$$

whence $\qquad A = \frac{1}{13}, \qquad B = -\frac{2}{39}, \qquad C = -\frac{6}{169}, \qquad E = \frac{62}{1521}.$

Therefore the general solution of the original equation is

$$y = c_1\sin x + c_2\cos x + e^{3x}\left(\tfrac{1}{13}x\sin x - \tfrac{2}{39}x\cos x - \tfrac{6}{169}\sin x + \tfrac{62}{1521}\cos x\right).$$

194. Systems of linear differential equations with constant coefficients. The operators of the previous articles may be employed in solving a system of two or more linear differential equations with constant coefficients, when the equations involve only one independent variable and a number of dependent variables equal to the number of the equations. The method by which this may be done can best be explained by an example.

Ex. $\dfrac{d^2x}{dt^2} - 3\dfrac{dy}{dt} + 4x = \sin 2t,$

$\dfrac{d^2y}{dt^2} + 3\dfrac{dx}{dt} + 4y = \cos 2t.$

These equations may be written

$$(D^2+4)x - 3Dy = \sin 2t, \tag{1}$$

$$3Dx + (D^2+4)y = \cos 2t. \tag{2}$$

We may now eliminate y from the equations in a manner analogous to that used in solving two algebraic equations. We first operate on (1) with D^2+4, the coefficient of y in (2), and have

$$(D^4+8D^2+16)x - 3(D^3+4D)y = 0, \tag{3}$$

since $(D^2+4)\sin 2t = -4\sin 2t + 4\sin 2t = 0$. We then operate on (2) with $3D$, the coefficient of y in (1), and have

$$9D^2x + 3(D^3+4D)y = -6\sin 2t, \tag{4}$$

since $3D\cos 2t = -6\sin 2t$. By adding (3) and (4) we have

$$(D^4+17D^2+16)x = -6\sin 2t, \tag{5}$$

the solution of which is

$$x = c_1\sin 4t + c_2\cos 4t + c_3\sin t + c_4\cos t + \tfrac{1}{6}\sin 2t. \tag{6}$$

Similarly, by operating on (1) with $3D$ and on (2) with D^2+4, and subtracting the result of the former operation from that of the latter, we have

$$(D^4+17D^2+16)y = -6\cos 2t, \tag{7}$$

the solution of which is

$$y = c_5\sin 4t + c_6\cos 4t + c_7\sin t + c_8\cos t + \tfrac{1}{6}\cos 2t. \tag{8}$$

The constants in (6) and (8) are, however, not all independent, for the values of x and y given in (6) and (8), if substituted in (1) and (2), must reduce the latter equations to identities. Making these substitutions, we have

$$12(c_6-c_1)\sin 4t - 12(c_5+c_2)\cos 4t + 3(c_8+c_3)\sin t \\ - 3(c_7-c_4)\cos t + \sin 2t = \sin 2t.$$

$$-12(c_6-c_1)\cos 4t - 12(c_5+c_2)\sin 4t + 3(c_8+c_3)\cos t \\ + 3(c_7-c_4)\sin t + \cos 2t = \cos 2t.$$

In order that these equations may be identically satisfied, we must have

$$c_6 = c_1, \quad c_5 = -c_2, \quad c_8 = -c_3, \quad c_7 = c_4.$$

Hence the solutions of (1) and (2) are

$$x = c_1 \sin 4t + c_2 \cos 4t + c_3 \sin t + c_4 \cos t + \tfrac{1}{6} \sin 2t, \tag{9}$$

$$y = -c_2 \sin 4t + c_1 \cos 4t + c_4 \sin t - c_3 \cos t + \tfrac{1}{6} \cos 2t. \tag{10}$$

The method of solving may be modified as follows: Having found as before the value of x in (6), we may substitute this value in (1). We have then

$$Dy = -4c_1 \sin 4t - 4c_2 \cos 4t + c_3 \sin t + c_4 \cos t - \tfrac{1}{3} \sin 2t,$$

whence $$y = c_1 \cos 4t - c_2 \sin 4t - c_3 \cos t + c_4 \sin t + \tfrac{1}{6} \cos 2t + C.$$

The constant C is found to be zero by substitution in (2), and we have again the solution (10).

195. The linear differential equation with variable coefficients. The equation

$$\frac{d^ny}{dx^n} + p_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1} \frac{dy}{dx} + p_n y = f(x), \tag{1}$$

where the coefficients $p_1, p_2, \cdots, p_{n-1}, p_n$ are functions of x, can rarely be solved in terms of elementary functions. In fact, such an equation usually defines a new transcendental function. We may, however, easily deduce certain simple properties of the solution of (1). Consider first the equation

$$\frac{d^ny}{dx^n} + p_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1} \frac{dy}{dx} + p_n y = 0, \tag{2}$$

which differs from (1) in that the right-hand side is zero.

If $y_1, y_2, \cdots, y_{n-1}, y_n$ *are* n *linearly independent* * *solutions of* (2), *then the general solution of* (2) *is*

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_{n-1} y_{n-1} + c_n y_n, \tag{3}$$

where $c_1, c_2, \cdots, c_{n-1}, c_n$ *are arbitrary constants.*

The fact that (3) is a solution of (2) may easily be verified by direct substitution in (2). That (3) is the general solution of (2) depends upon the fact that it contains n arbitrary constants, and the number of constants in the general solution of a differential equation is equal to the order of the equation. This statement we shall not prove.

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* The n functions $y_1, y_2, \cdots, y_{n-1}, y_n$ are said to be linearly independent if there exists no relation of the form

$$a_1 y_1 + a_2 y_2 + \cdots + a_{n-1} y_{n-1} + a_n y_n \equiv 0,$$

where $a_1, a_2, \cdots, a_{n-1}, a_n$ are constants, and $\equiv$ means "identically equal."

Returning now to (1), we may say:

If $y_1, y_2, \cdots, y_{n-1}, y_n$ *are* n *linearly independent solutions of* (2), *and* u *is any particular solution of* (1), *then the general solution of* (1) *is*

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n + u, \tag{4}$$

where $c_1, c_2, \cdots, c_n$ *are arbitrary constants.*

The fact that (4) is a solution of (1) may be verified by substitution. The fact that it is the general solution we shall accept without proof.

It appears now that the complementary function and the particular integral of § 192 are only special cases of (4). There exists, however, no general method of finding the solution (4) when the coefficients of (1) are not constant.

Methods of solution may, however, exist in special cases, and we shall notice especially the equation

$$x^n\frac{d^ny}{dx^n} + a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}x\frac{dy}{dx} + a_ny = f(x), \tag{5}$$

where $a_1, a_2, \cdots, a_{n-1}, a_n$ are constants. This equation has the peculiarity that each derivative is multiplied by a power of x equal to the order of the derivative. It can be reduced to a linear equation with constant coefficients, by placing

$$x = e^z.$$

For
$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = e^{-z}\frac{dy}{dz} = e^{-z}Dy,$$

$$\frac{d^2y}{dx^2} = e^{-z}\frac{d}{dz}\left(e^{-z}\frac{dy}{dz}\right) = e^{-2z}\frac{d^2y}{dz^2} - e^{-2z}\frac{dy}{dz}$$
$$= e^{-2z}(D^2 - D)y,$$

$$\frac{d^3y}{dx^3} = e^{-z}\frac{d}{dz}\left(e^{-2z}\frac{d^2y}{dz^2}\right) - e^{-z}\frac{d}{dz}\left(e^{-2z}\frac{dy}{dz}\right)$$
$$= e^{-3z}\frac{d^3y}{dz^3} - 3e^{-3z}\frac{d^2y}{dz^2} + 2e^{-3z}\frac{dy}{dz}$$
$$= e^{-3z}(D^3 - 3D^2 + 2D)y,$$

. . . . . .

where
$$D = \frac{d}{dz}.$$

Hence
$$x\frac{dy}{dx}=Dy,$$
$$x^2\frac{d^2y}{dx^2}=(D^2-D)y,$$
$$x^3\frac{d^3y}{dx^3}=(D^3-3D^2+2D)y.$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Ex. $x^3\frac{d^3y}{dx^3}+5x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}=x^2.$

Placing $x=e^z$, and making the substitution as above, we have
$$(D^3+2D^2)y=e^{2z},$$
whence $y=c_1+c_2z+c_3e^{-2z}+\frac{1}{16}e^{2z}=c_1+c_2(\log x)+\frac{c_3}{x^2}+\frac{1}{16}x^2.$

196. Solution by series. The solution of a linear differential equation can usually be expanded into a Taylor's or a Maclaurin's series. This is, in fact, an important and powerful method of investigating the function defined by the equation. We shall limit ourselves, however, to showing by examples how the series may be obtained. The method consists in assuming a series of the form
$$y=a_0x^m+a_1x^{m+1}+a_2x^{m+2}+\cdots,$$
where m and the coefficients a_0, a_1, a_2, $\cdots$ are undetermined. This series is then substituted in the differential equation, and m and the coefficients are so determined that the equation is identically satisfied.

Ex. 1. $x\frac{d^2y}{dx^2}+(x-3)\frac{dy}{dx}-2y=0.$

We assume a series of the form given above, and write the expression for each term of the differential equation, placing like powers of x under each other. We have then
$$\begin{aligned}
x\frac{d^2y}{dx^2} &= m(m-1)a_0x^{m-1}+(m+1)ma_1x^m+\cdots+(m+r+1)(m+r)a_{r+1}x^{m+r}+\cdots,\\
x\frac{dy}{dx} &= \qquad\qquad ma_0x^m+\cdots \qquad +(m+r)a_rx^{m+r}+\cdots,\\
-3\frac{dy}{dx} &= -3ma_0x^{m-1}-3(m+1)a_1x^m-\cdots -3(m+r+1)a_{r+1}x^{m+r}-\cdots,\\
-2y &= \qquad\qquad -2a_0x^m-\cdots \qquad -2a_rx^{m+r}-\cdots.
\end{aligned}$$
Adding these results, we have an expression which must be identically equal to zero, since the assumed series satisfies the differential equation. Equating to zero the coefficient of x^{m-1}, we have
$$m(m-4)a_0=0. \tag{1}$$

Equating to zero the coefficient of x^m, we have

$$(m+1)(m-3)a_1 + (m-2)a_0 = 0. \tag{2}$$

Finally, equating to zero the coefficient of x^{m+r}, we have the more general relation

$$(m+r+1)(m+r-3)a_{r+1} + (m+r-2)a_r = 0. \tag{3}$$

We shall gain nothing by placing $a_0 = 0$ in equation (1), since a_0x^m is assumed as the first term of the series. Hence to satisfy (1) we must have either

$$m = 0 \quad \text{or} \quad m = 4.$$

Taking the first of these possibilities, namely $m = 0$, we have, from (2),

$$a_1 = -\tfrac{2}{3}a_0,$$

and from (3),

$$a_{r+1} = -\frac{r-2}{(r+1)(r-3)}a_r. \tag{4}$$

This last formula (4) enables us to compute any coefficient, a_{r+1}, when we know the previous one, a_r. Thus we find $a_2 = -\frac{1}{4}a_1 = \frac{1}{6}a_0$, $a_3 = 0$, and therefore all coefficients after a_3 equal to zero.

Hence we have as one solution of the differential equation the polynomial

$$y_1 = a_0(1 - \tfrac{2}{3}x + \tfrac{1}{6}x^2). \tag{5}$$

Returning now to the second of the two possibilities for the value of m, we take $m = 4$. Then (2) becomes

$$5a_1 + 2a_0 = 0,$$

and (3) becomes

$$a_{r+1} = -\frac{r+2}{(r+5)(r+1)}a_r. \tag{6}$$

Computing from this the coefficients of the first four terms of the series, we have the solution

$$y_2 = a_0\left(x^4 - \frac{2}{5}x^5 + \frac{3}{5\cdot 6}x^6 - \frac{4}{5\cdot 6\cdot 7}x^7 + \cdots\right). \tag{7}$$

We have now in (5) and (7) two independent solutions of the differential equation. Hence, by § 195, the general solution is

$$y = c_1y_1 + c_2y_2.$$

Ex. 2. *Legendre's equation.* $(1-x^2)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + n(n+1)y = 0.$

Assuming the general form of the series, we have

$$\begin{aligned}
\frac{d^2y}{dx^2} &= m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1} + (m+2)(m+1)a_2x^m + \cdots,\\
-x^2\frac{d^2y}{dx^2} &= \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -m(m-1)a_0x^m - \cdots,\\
-2x\frac{dy}{dx} &= \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad -2ma_0x^m - \cdots,\\
n(n+1)y &= \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n(n+1)a_0x^m + \cdots.
\end{aligned}$$

Equating to zero the coefficients of x^{m-2}, x^{m-1}, and x^m, we have

$$m(m-1)a_0 = 0, \tag{1}$$

$$(m+1)ma_1 = 0, \tag{2}$$

$$(m+2)(m+1)a_2 - (m-n)(m+n+1)a_0 = 0. \tag{3}$$

To find a general law for the coefficients, we will find the term containing x^{m+r-2} in each of the above expansions, this term being chosen because it contains a_r in the first expansion. We have

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \cdots + (m+r)(m+r-1)a_r x^{m+r-2} + \cdots,\\
-x^2\frac{d^2y}{dx^2} &= \cdots - (m+r-2)(m+r-3)a_{r-2}x^{m+r-2} - \cdots,\\
-2x\frac{dy}{dx} &= \cdots -2(m+r-2)a_{r-2}x^{m+r-2} - \cdots,\\
n(n+1)y &= \cdots + n(n+1)a_{r-2}x^{m+r-2} + \cdots.
\end{aligned}$$

The sum of these coefficients equated to zero gives

$$(m+r)(m+r-1)a_r - (m-n+r-2)(m+n+r-1)a_{r-2} = 0. \tag{4}$$

We may satisfy (1) either by placing $m = 0$ or by placing $m = 1$. We shall take $m = 0$. Then, from (2), a_1 is arbitrary; from (3)

$$a_2 = -\frac{n(n+1)}{2}a_0; \tag{5}$$

and from (4)

$$a_r = -\frac{(n-r+2)(n+r-1)}{r(r-1)}a_{r-2}. \tag{6}$$

By means of (6) we determine the solution

$$\begin{aligned}
y = a_0\left(1 - \frac{n(n+1)}{\lfloor 2}x^2 + \frac{n(n-2)(n+1)(n+3)}{\lfloor 4}x^4 - \cdots\right)&\\
+ a_1\left(x - \frac{(n-1)(n+2)}{\lfloor 3}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{\lfloor 5}x^5 - \cdots\right)&.
\end{aligned} \tag{7}$$

Since a_0 and a_1 are arbitrary, we have in (7) the general solution of the differential equation. In fact, the student will find that if he takes the value $m = 1$ from (1), he will obtain again the second series in (7).

Particular interest attaches to the cases in which one of the series in (7) reduces to a polynomial. This evidently happens to the first series when n is an even integer, and to the second series when n is an odd integer. By giving to a_0 or a_1 such numerical values in each case that the polynomial is equal to unity when x equals unity, we obtain from the series in (7) the polynomials

$$\begin{aligned}
P_1 &= x,\\
P_2 &= \frac{3}{2}x^2 - \frac{1}{2},\\
P_3 &= \frac{5}{2}x^3 - \frac{3}{2}x,\\
P_4 &= \frac{7\cdot 5}{4\cdot 2}x^4 - 2\,\frac{5\cdot 3}{4\cdot 2}x^2 + \frac{3\cdot 1}{4\cdot 2},\\
P_5 &= \frac{9\cdot 7}{4\cdot 2}x^5 - 2\,\frac{7\cdot 5}{4\cdot 2}x^3 + \frac{5\cdot 3}{4\cdot 2}x,
\end{aligned}$$

each of which satisfies a Legendre's differential equation in which n has the value indicated by the suffix of P. These polynomials are called *Legendre's coefficients.*

Ex. 3. *Bessel's equation.* $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (x^2 - n^2)\,y = 0.$

Assuming the series for y in the usual form, we have

$$\begin{aligned}
x^2\frac{d^2y}{dx^2} &= m(m-1)\,a_0x^m + (m+1)\,ma_1x^{m+1} + (m+2)(m+1)\,a_2x^{m+2} + \cdots,\\
x\frac{dy}{dx} &= ma_0x^m + (m+1)\,a_1x^{m+1} + (m+2)\,a_2x^{m+2} + \cdots,\\
-n^2y &= -n^2a_0x^m - n^2a_1x^{m+1} - n^2a_2x^{m+2} - \cdots,\\
x^2y &= a_0x^{m+2} + \cdots.
\end{aligned}$$

Equating to zero the sum of the coefficients of the first three powers of x, we have

$$(m^2 - n^2)\,a_0 = 0, \tag{1}$$

$$[(m+1)^2 - n^2]a_1 = 0, \tag{2}$$

$$[(m+2)^2 - n^2]a_2 + a_0 = 0. \tag{3}$$

To obtain the general expression for the coefficients, we have

$$\begin{aligned}
x^2\frac{d^2y}{dx^2} &= \cdots + (m+r)(m+r-1)\,a_rx^{m+r} + \cdots,\\
x\frac{dy}{dx} &= \cdots + (m+r)\,a_rx^{m+r} + \cdots,\\
-n^2y &= \cdots - n^2a_rx^{m+r} - \cdots,\\
x^2y &= \cdots + a_{r-2}x^{m+r} + \cdots.
\end{aligned}$$

Equating to zero the sum of these coefficients, we have

$$[(m+r)^2 - n^2]a_r + a_{r-2} = 0. \tag{4}$$

Equation (1) may be satisfied by $m = \pm n$. We will take first $m = n$. Then from (2), (3), and (4) we have

$$a_1 = 0,\quad a_2 = -\frac{a_0}{2\,(2\,n+2)},\quad a_r = -\frac{a_{r-2}}{r\,(2\,n+r)}.$$

By use of these results we obtain the series

$$y_1 = a_0 x^n\left(1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2\cdot 4\cdot(2n+2)(2n+4)} - \cdots\right). \tag{5}$$

Similarly, by placing $m = -n$, we obtain the series

$$y_2 = a_0 x^{-n}\left(1 + \frac{x^2}{2(2n-2)} + \frac{x^4}{2\cdot 4\cdot(2n-2)(2n-4)} + \cdots\right). \tag{6}$$

If, now, n is any number except an integer or zero, each of the series (5) and (6) converges and the two series are distinct from each other. Hence in this case the general solution of the differential equation is

$$y = c_1 y_1 + c_2 y_2.$$

If $n = 0$, the two series (5) and (6) are identical. If n is a positive integer, series (6) is meaningless, since some of the coefficients become infinite. If n is a negative integer, series (5) is meaningless, since some of the coefficients become infinite. Hence, if n is zero or an integer, we have in (5) and (6) only one particular solution of the differential equation, and another particular solution must be found before the general solution is known. The manner in which this may be done cannot, however, be taken up here.

The series (5) and (6) define new transcendental functions of x called *Bessel's functions.* They are important in many applications to mathematical physics.

PROBLEMS

Solve the following equations:

1. $3\frac{dy}{dx} + 2y = 0.$
2. $\frac{dy}{dx} + 3y = x^2 + \frac{1}{2}\sin x.$
3. $\frac{dy}{dx} - 2y = e^{3x} + e^x \cos x.$
4. $\frac{dy}{dx} + y = 3e^{-x} + xe^x.$
5. $\frac{dy}{dx} + 4y = 6\sin^2 x.$
6. $\frac{dy}{dx} - y = \frac{2}{e^x + e^{-x}}.$
7. $\frac{dy}{dx} - 2y = \sin 2x \cos 3x.$
8. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 8y = 0.$
9. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} = 0.$
10. $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = 0.$
11. $\frac{d^2y}{dx^2} + 9y = 0.$
12. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0.$
13. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = x^4 - 2x^3 + 5x.$
14. $\frac{d^2y}{dx^2} - y = 4\sin^2 x.$
15. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 10y = 2\cos 3x.$
16. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 4y = 6.$
17. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = x^3 + 4x^2 + 1.$
18. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = e^{2x}(5x^2 - 8x - 4).$

19. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = e^x + 4e^{2x}.$

20. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = x^3 - 1.$

21. $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = \dfrac{e^{3x}}{(x-3)^2}.$

22. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 5e^{2x}\sin 3x.$

23. $\dfrac{d^2y}{dx^2} + 4y = e^{2x}\sin 2x.$

24. $\dfrac{d^2y}{dx^2} + y = 2\sin 5x \sin 3x.$

25. $\dfrac{d^2y}{dx^2} + 4y = 4\cos 2x.$

26. $\dfrac{d^2y}{dx^2} + 3y = x\sin^2 x.$

27. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 13y = e^{2x}\sin 3x.$

28. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 1 + e^x.$

29. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = x^2e^{-x} + \frac{1}{3}e^{3x}.$

30. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + y = 3x^2 + 4x - 1.$

31. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} = 0.$

32. $\dfrac{d^3y}{dx^3} + \dfrac{dy}{dx} = 0.$

33. $\dfrac{d^3y}{dx^3} - y = 0.$

34. $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = 0.$

35. $\dfrac{d^4y}{dx^4} + \dfrac{d^2y}{dx^2} = 2.$

36. $\dfrac{d^4y}{dx^4} + y = \frac{1}{2}e^x - \frac{1}{2}e^{-x}.$

37. $\dfrac{d^3y}{dx^3} + y = (e^x + e^{-x})\cos x.$

38. $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + 49y = 3x^3 - \frac{1}{2}e^{-3x}.$

39. $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} = (x^2+1)e^x.$

40. $\dfrac{d^4y}{dx^4} + 8\dfrac{d^2y}{dx^2} + 16y = \frac{1}{2}\sin 2x.$

41. $\dfrac{d^3y}{dx^3} - 4\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} = 3e^{2x} + 4.$

42. $\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} = x^3.$

43. $\dfrac{dx}{dt} + \dfrac{dy}{dt} - y = 0,$

$\dfrac{dx}{dt} + \dfrac{dy}{dt} + 2x - 3y = 0.$

44. $\dfrac{dx}{dt} + \dfrac{dy}{dt} - x - 4y = e^{5t},$

$\dfrac{dx}{dt} + \dfrac{dy}{dt} - 2x - 3y = e^{2t}.$

45. $2\dfrac{dx}{dt} + \dfrac{dy}{dt} + 3x = e^t,$

$\dfrac{dx}{dt} + \dfrac{dy}{dt} - 2y = \sin 2t.$

46. $\dfrac{dx}{dt} = 2x - y - 5,$

$\dfrac{dy}{dt} = 3y - 2x + 4.$

47. $\dfrac{dx}{dt} = 5x - y - 13,$

$\dfrac{dy}{dt} = 2x + 2y - 10.$

48. $\dfrac{d^2x}{dt^2} + \dfrac{dy}{dt} = t^2,$

$\dfrac{d^2x}{dt^2} + 2\dfrac{dy}{dt} + 2x - y = 2t.$

49. $\dfrac{d^2x}{dt^2} - 2\dfrac{dx}{dt} + \dfrac{dy}{dt} = 3e^{-2t},$

$\dfrac{dx}{dt} + \dfrac{dy}{dt} - 2x + 2y = t^3.$

50. $\dfrac{d^2x}{dt^2} - a^2y = 0,$

$\dfrac{d^2y}{dt^2} + a^2x = 0.$

51. $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = x^3.$

52. $x^2\dfrac{d^2y}{dx^2} + 7x\dfrac{dy}{dx} + 13y = x\log x.$

53. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + 4y = (\log x)^2.$

54. $x^3\dfrac{d^3y}{dx^3} + 3x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = x^2.$

55. $x^3\dfrac{d^3y}{dx^3} + 2x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = \dfrac{1}{x}.$

56. $x^3\dfrac{d^3y}{dx^3} + 3x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} = \log x^3.$

Solve the following equations by means of series:

57. $x^2\dfrac{d^2y}{dx^2} + (x - 2x^2)\dfrac{dy}{dx} - 9y = 0.$

58. $x\dfrac{d^2y}{dx^2} + (x - 3)\dfrac{dy}{dx} - 2y = 0.$

59. $(x - x^2)\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y = 0.$

60. $x^2\dfrac{d^2y}{dx^2} + x^2\dfrac{dy}{dx} + (x - 6)y = 0.$

61. $\dfrac{d^2y}{dx^2} + nxy = 0.$

62. $(1 + x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - ny = 0.$

63. A particle of unit mass moving in a straight line is acted on by an attracting force in its line of motion directed toward a center and proportional to the distance of the particle from the center, and also by a periodic force equal to $a\cos kt$. Determine its motion.

64. A particle of unit mass moving in a straight line is acted on by three forces, — an attracting force in its line of motion directed toward a center and proportional to the distance of the particle from the center, a resisting force proportional to the velocity of the particle, and a periodic force equal to $a\cos kt$. Determine the motion of the particle.

65. Under what conditions will the motion of the particle in Ex. 64 consist of oscillations the amplitudes of which increase without limit as the time increases without limit?

CHAPTER XIX

PARTIAL DIFFERENTIAL EQUATIONS

197. Introduction. A *partial differential equation* is an equation which involves partial derivatives. The general solution of such an equation involves one or more arbitrary functions. Thus $z = f(x-y,\ y-x)$ is a solution of the equation $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$ (§ 113, Ex. 1), no matter what is the form of the function f. Also $z = f_1(x+at) + f_2(x-at)$ is a solution of the equation $\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}$ (§ 118, Ex. 2), no matter what are the forms of the functions f_1 and f_2.

Only in comparatively few cases can the solution of a partial differential equation be written down explicitly. In general, the nature and the properties of functions defined by such equations must be studied by the methods of advanced mathematics. In a practical application, the problem is usually to determine a function which will satisfy the differential equation and at the same time meet the other conditions of the practical problem.

198. Special forms of partial differential equations. Partial differential equations sometimes occur which can be readily solved by successive integration with respect to each of the variables, or which can be otherwise solved by elementary methods. No general discussion can very well be given for such equations, but the following examples will illustrate them.

Ex. 1. $\frac{\partial^2 z}{\partial x \partial y} = 0.$

By integration with respect to y, we have

$$\frac{\partial z}{\partial x} = \phi_1(x),$$

where ϕ_1 is an arbitrary function. Integrating with respect to x, we have

$$z = \phi_2(x) + \phi_3(y),$$

where both ϕ_2 and ϕ_3 are arbitrary functions.

Ex. 2. $\frac{\partial^2 z}{\partial x^2} = - a^2 z.$

If x were the only independent variable, the solution of this equation would be

$$z = c_1 \sin ax + c_2 \cos ax.$$

This solution will also hold for the partial differential equation if we simply impose upon c_1 and c_2 the condition to be independent of x but not necessarily independent of the other variables. That is, if z is a function of x and y, we have for the solution of the differential equation

$$z = \phi_1(y) \sin ax + \phi_2(y) \cos ax,$$

where $\phi_1(y)$ and $\phi_2(y)$ are arbitrary functions.

Ex. 3. $\frac{\partial^2 z}{\partial y^2} - a^2 \frac{\partial^2 z}{\partial x^2} = 0.$

Placing $x + ay = u$, and $x - ay = v$, we have (§ 118)

$$\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial u^2} - 2 a^2 \frac{\partial^2 z}{\partial u \partial v} + a^2 \frac{\partial^2 z}{\partial v^2},$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial u^2} + 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2},$$

and the differential equation becomes

$$\frac{\partial^2 z}{\partial u \partial v} = 0,$$

the solution of which (Ex. 1) is

$$z = \phi_1(u) + \phi_2(v).$$

Hence the solution of the given equation is

$$z = \phi_1(x + ay) + \phi_2(x - ay).$$

When $a^2 = -1$, we have $z = \phi_1(x + iy) + \phi_2(x - iy)$

as the solution of the equation $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$

199. The linear partial differential equation of the first order. Consider the equation

$$P \frac{\partial z}{\partial x} + Q \frac{\partial z}{\partial y} = R, \tag{1}$$

where P, Q, and R are constants or functions of one or more of the variables x, y, and z. The solution of (1) is a function

$$z = f(x, y), \tag{2}$$

which, substituted in (1), reduces it to an identity.

Now equation (2) represents a surface, the normal at any point of which has the direction $\frac{\partial z}{\partial x}:\frac{\partial z}{\partial y}:-1$ (§ 112). Equation (1) therefore asserts that the normal to (2) at any point is perpendicular to the direction $P:Q:R$ (§ 98, (5)). Consequently we may start from any point on (2) and, moving in the direction $P:Q:R$, remain always on the surface. That is, the surface (2) is covered by a family of curves each of which is a solution of the simultaneous equations

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}. \tag{3}$$

Now (2) is any solution of (1), and hence we reach the conclusion that the solution of (1) consists of all surfaces which are covered by a family of curves each of which is a solution of (3).

We may proceed to find these surfaces as follows: Let us solve (3), obtaining, as in § 185, the solution

$$u(x, y, z)=c_1, \qquad v(x, y, z)=c_2. \tag{4}$$

Then, if we form the equation

$$\phi(u, v)=0, \tag{5}$$

where ϕ is any function whatever, we have a surface which is covered by curves represented by (4). For if in (4) we give c_1 and c_2 such values that $\phi(c_1, c_2)=0$, the corresponding curve lies on (5). That is, by means of (5) we have assembled the curves (4) into surfaces, and have therefore the solution of (1). We may formulate our result into the following rule:

To solve the equation $P\frac{\partial z}{\partial x}+Q\frac{\partial z}{\partial y}=R,$

solve first the equations $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

for the solution $u=c_1, \quad v=c_2.$

Then the equation $\phi(u, v)=0,$

where ϕ *is an arbitrary function, is the solution of the partial differential equation.*

Ex. $$(ny - mz)\frac{\partial z}{\partial x} + (lz - nx)\frac{\partial z}{\partial y} = mx - ly. \tag{1}$$

We form the equations $$\frac{dx}{ny - mz} = \frac{dy}{lz - nx} = \frac{dz}{mx - ly}, \tag{2}$$
the solutions of which are

$$x^2 + y^2 + z^2 = c_1, \qquad lx + my + nz = c_2. \tag{3}$$

Hence the solution of (1) is

$$\phi(x^2 + y^2 + z^2, lx + my + nz) = 0. \tag{4}$$

Geometrically, the first equation of (3) represents all spheres with their centers at the origin, and the second equation of (3) represents all planes which are normal to the line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}. \tag{5}$$

Hence the two equations (3) taken simultaneously represent all circles whose centers are in the line (5) and whose planes are perpendicular to (5). Equation (4), then, represents all surfaces which can be formed out of these circles; that is, all surfaces of revolution which have the line (5) for an axis. These surfaces of revolution form the solution of (1).

200. Laplace's equation in the plane. Solutions of Laplace's equation in the plane,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \tag{1}$$

have already been found in §§ 172, 198. Another method of dealing with this equation is as follows: Let us place $V = XY$, where X is a function of x alone and Y is a function of y alone, and ask if it is possible to determine X and Y so that Laplace's equation may be satisfied. Substituting in the given equation and dividing by XY, we have

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0,$$

which may be put in the form

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2}. \tag{2}$$

According to the hypothesis, the left-hand member of (2) cannot contain y and the right-hand member cannot contain x. Hence they are each equal to some constant, which we will denote by α^2. Then (2) breaks up into two ordinary differential equations,

$$\frac{d^2X}{dx^2} - \alpha^2 X = 0, \tag{3}$$

$$\frac{d^2Y}{dy^2} + \alpha^2 Y = 0. \tag{4}$$

By § 190, the solution of (3) is

$$X = A_1 e^{\alpha x} + A_2 e^{-\alpha x},$$

and the solution of (4) is

$$Y = B_1 \cos \alpha y + B_2 \sin \alpha y.$$

Hence $\quad V = e^{\alpha x} \cos \alpha y, \quad V = e^{\alpha x} \sin \alpha y,$

$$V = e^{-\alpha x} \cos \alpha y, \quad V = e^{-\alpha x} \sin \alpha y,$$

are particular solutions of the given equation.

If the value of the constant had been denoted by $-\alpha^2$, we should have obtained the particular solutions

$$V = e^{\alpha y} \cos \alpha x, \qquad V = e^{\alpha y} \sin \alpha x,$$

$$V = e^{-\alpha y} \cos \alpha x, \quad V = e^{-\alpha y} \sin \alpha x.$$

The particular solutions thus obtained are, in fact, the real and the imaginary parts of the functions $e^{\alpha z}$, $e^{-\alpha z}$, $e^{-\iota\alpha z}$, and $e^{\iota\alpha z}$ (§ 172).

The sum of two or more particular solutions of (1), each multiplied by an arbitrary constant, is also a solution of (1), as is easily verified. Hence we may form the particular solutions

$$V = A_0 + \sum_{m=1}^{m=\infty} A_m e^{-my} \sin mx + \sum_{m=1}^{m=\infty} B_m e^{-my} \cos mx, \tag{5}$$

$$V = A_0 + \sum_{m=1}^{m=\infty} A_m e^{my} \sin mx + \sum_{m=1}^{m=\infty} B_m e^{my} \cos mx. \tag{6}$$

Solution (5) has the property of reducing to A_0 when $y = \infty$, while solution (6) becomes infinite with y.

Ex. Find the permanent temperature at any point of a thin rectangular plate of breadth π and of infinite length, the end being kept at the temperature unity and the long edges being kept at the temperature zero.

If u is the temperature, it is known that u satisfies the differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{1}$$

If we take the end of the plate as the axis of x, and one of the long edges as the axis of y, we have to solve (1) subject to the conditions:

$$\text{if}\quad x = 0, \qquad u = 0, \tag{2}$$

$$\text{if}\quad x = \pi, \qquad u = 0, \tag{3}$$

$$\text{if}\quad y = \infty, \qquad u = 0, \tag{4}$$

$$\text{if}\quad y = 0, \qquad u = 1. \tag{5}$$

Condition (4) is satisfied by the solution

$$u = \sum_{m=1}^{m=\infty} A_m e^{-my} \sin mx + \sum_{m=1}^{m=\infty} B_m e^{-my} \cos mx.$$

By condition (2),

$$0 = \sum_{m=1}^{m=\infty} B_m e^{-my}$$

for all values of y, and hence $B_m = 0$ for all values of m. Our solution is now reduced to

$$u = \sum_{m=1}^{m=\infty} A_m e^{-my} \sin mx,$$

which satisfies (2), (3), and (4). In order that it may satisfy (5), we must so determine the coefficients A_m that

$$1 = \sum_{m=1}^{m=\infty} A_m \sin mx. \tag{6}$$

But (6) is a special form of a Fourier's series (§§ 159, 160). Accordingly, we multiply (6) by $\sin mx\, dx$, and integrate from 0 to π. As a result, we have $A_m = \frac{2}{\pi}\left(\frac{1 - \cos m\pi}{m}\right)$, whence $A_1 = \frac{4}{\pi}$, $A_2 = 0$, $A_3 = \frac{4}{\pi} \cdot \frac{1}{3}$, $\cdots$. Therefore

$$u = \frac{4}{\pi}\left(e^{-y} \sin x + \frac{1}{3} e^{-3y} \sin 3x + \frac{1}{5} e^{-5y} \sin 5x + \cdots\right)$$

is the solution of our problem, since it satisfies all the conditions.

201. Laplace's equation in three dimensions. The general form of Laplace's equation in rectangular coördinates is

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \tag{1}$$

If cylindrical coördinates are used, (1) becomes

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0, \tag{2}$$

and in polar coördinates (1) becomes

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi \frac{\partial V}{\partial\phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2 V}{\partial\theta^2} = 0. \tag{3}$$

The general discussion of these equations is beyond the limits of this text. We shall, however, consider particular solutions of (3) in the special case in which V is independent of the diedral angle θ. In that case $\frac{\partial^2 V}{\partial \theta^2} = 0$, and (3) reduces to

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial V}{\partial\phi}\right) = 0. \tag{4}$$

Letting $V = R\Phi$, where R is a function of r alone and Φ is a function of ϕ alone, we may replace (4) by the two ordinary differential equations

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \alpha^2 R = 0, \tag{5}$$

$$\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{d\Phi}{d\phi}\right) + \alpha^2\Phi = 0, \tag{6}$$

where α^2 is an arbitrary constant.

Expanding (5), we have

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - \alpha^2 R = 0,$$

the solution of which is, by § 195,

$$R = A_1 r^{-\frac{1}{2}+\sqrt{\alpha^2+\frac{1}{4}}} + A_2 r^{-\frac{1}{2}-\sqrt{\alpha^2+\frac{1}{4}}} = A_1 r^m + \frac{A_2}{r^{m+1}},$$

where $m = -\frac{1}{2} + \sqrt{\alpha^2 + \frac{1}{4}}$. From this value of m we have $\alpha^2 = m(m+1)$, and (6) becomes

$$\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{d\Phi}{d\phi}\right) + m(m+1)\Phi = 0. \tag{7}$$

Changing the independent variable from ϕ to t, where $t = \cos\phi$, we have Legendre's equation (§ 196, Ex. 2),

$$(1-t^2)\frac{d^2\Phi}{dt^2} - 2t\frac{d\Phi}{dt} + m(m+1)\Phi = 0. \tag{8}$$

In the particular case in which m is an integer, we may choose for Φ Legendre's coefficient $P_m(t) = P_m(\cos\phi)$.

Therefore the particular solutions of (4) are

$$V = r^m P_m(\cos\phi),$$

$$V = \frac{P_m(\cos\phi)}{r^{m+1}}.$$

Ex. Find the potential due to a circular ring of small cross section and radius a.

If the center of the ring is taken as the origin of coördinates, and the axis OZ is perpendicular to the plane of the ring, Laplace's equation assumes the form

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial V}{\partial\phi}\right)=0, \qquad (1)$$

since, from the symmetry of the problem, V is independent of θ.

This equation is satisfied by

$$V=\sum_{m=0}^{m=\infty}\left(A_m r^m+\frac{B_m}{r^{m+1}}\right)P_m(\cos\phi), \qquad (2)$$

where A_m and B_m are arbitrary constants.

At any point on the axis OZ distant r from the origin

$$V=\frac{M}{\sqrt{a^2+r^2}}, \qquad (3)$$

where M is the mass of the ring, as shown by the method of Ex. 9, p. 99.

Then, when $\phi=0$, $\cos\phi=1$, and by § 196, Ex. 2, $P_m(\cos\phi)=1$. At the same time the right-hand members of (2) and (3) must be equal, i.e.

$$\frac{M}{\sqrt{a^2+r^2}}=\sum_{m=0}^{m=\infty}\left(A_m r^m+\frac{B_m}{r^{m+1}}\right),$$

and the coefficients A_m and B_m must be chosen so as to satisfy this equation. If $r<a$ (§ 31, Ex. 4),

$$\frac{M}{\sqrt{a^2+r^2}}=\frac{M}{a}\left(1-\frac{1}{2}\frac{r^2}{a^2}+\frac{1\cdot 3}{2\cdot 4}\frac{r^4}{a^4}-\cdots\right),$$

and if $r>a$,

$$\frac{M}{\sqrt{a^2+r^2}}=\frac{M}{a}\left(\frac{a}{r}-\frac{1}{2}\frac{a^3}{r^3}+\frac{1\cdot 3}{2\cdot 4}\frac{a^5}{r^5}-\cdots\right).$$

Hence if $r<a$, we place all the B's equal to zero, $A_0=\frac{M}{a}$, $A_1=0$, $A_2=-\frac{M}{a}\cdot\frac{1}{2}\cdot\frac{1}{a^2}$, etc., and obtain the solution

$$V=\frac{M}{a}\left(1-\frac{1}{2}\frac{r^2}{a^2}P_2(\cos\phi)+\frac{1\cdot 3}{2\cdot 4}\frac{r^4}{a^4}P_4(\cos\phi)-\cdots\right),$$

and if $r>a$, we place all the A's equal to zero, $B_0=\frac{M}{a}\cdot a$, $B_1=0$, $B_2=-\frac{M}{a}\cdot\frac{1}{2}\cdot a^3$, $\cdots$, and obtain the solution

$$V=\frac{M}{a}\left(\frac{a}{r}P_0(\cos\phi)-\frac{1}{2}\cdot\frac{a^3}{r^3}P_2(\cos\phi)+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{a^5}{r^5}P_4(\cos\phi)-\cdots\right).$$

PROBLEMS

Solve the following equations:

1. $\frac{\partial^2 z}{\partial x^2}=a^2 z.$

2. $x\frac{\partial^2 z}{\partial x^2}=a\frac{\partial z}{\partial x}.$

3. $\frac{\partial^2 z}{\partial x^2}-a\frac{\partial z}{\partial x}=0.$

4. $\frac{\partial^2 z}{\partial x^2}-5\frac{\partial z}{\partial x}+6z=0.$

5. $\frac{\partial^2 z}{\partial y^2}+4\frac{\partial z}{\partial y}+5z=0.$

6. $\frac{\partial^2 z}{\partial x\partial y}=xy.$

7. $\dfrac{\partial^2 z}{\partial x \partial y} = x^2 + y^2.$

8. $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 1.$

9. $x\dfrac{\partial z}{\partial y} - y\dfrac{\partial z}{\partial x} = 0.$

10. $2\,zx\dfrac{\partial z}{\partial x} + 2\,zy\dfrac{\partial z}{\partial y} = z^2.$

11. $(z + x)\dfrac{\partial z}{\partial y} - (z + y)\dfrac{\partial z}{\partial x} = y - x.$

12. $y\dfrac{\partial z}{\partial x} - x\dfrac{\partial z}{\partial y} = 1.$

13. $(7x + y)\dfrac{\partial z}{\partial x} + (8x + 5y)\dfrac{\partial z}{\partial y} = 1.$

14. $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z.$

15. $x^2\dfrac{\partial z}{\partial x} - xy\dfrac{\partial z}{\partial y} + y^2 = 0.$

16. $xz\dfrac{\partial z}{\partial x} + yz\dfrac{\partial z}{\partial y} = xy.$

17. $yz\dfrac{\partial z}{\partial x} + zx\dfrac{\partial z}{\partial y} = xy.$

18. $y\dfrac{\partial z}{\partial x} + (x + z)\dfrac{\partial z}{\partial y} = y.$

19. Find particular solutions of the equation $\dfrac{\partial^2 y}{\partial t^2} = a^2\dfrac{\partial^2 y}{\partial x^2}$ in terms of trigonometric functions of x and t.

20. Find particular solutions of Laplace's equation in the plane in polar coördinates $\dfrac{\partial^2 V}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial V}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 V}{\partial \theta^2} = 0.$

21. Find the permanent temperature at any point in a semicircular plate of radius unity, the circumference of which is kept at the temperature unity and the bounding diameter of which is kept at the temperature 0, given that the temperature u satisfies the differential equation of Ex. 20.

22. The equation for the linear flow of heat is $\dfrac{\partial u}{\partial t} = a^2\dfrac{\partial^2 u}{\partial x^2}$, where u is the temperature at any time t, and x is measured parallel to the direction in which the heat flows. If a slab of thickness π is originally at the temperature unity throughout, and both faces are then kept at a temperature 0, find the temperature at any point of the slab, the slab being so large that only the flow of heat normal to its bounding faces need be considered.

ANSWERS

(The answers to some problems are intentionally omitted.)

CHAPTER II

Page 33

1. $x^3 + 3x^2 + x.$

2. $\frac{1}{3}x^3 + 2x - \frac{1}{x}.$

3. $\frac{2\sqrt{x^5}}{5} - \frac{2}{\sqrt{x}}.$

4. $\frac{2}{3}x^{\frac{3}{2}} + \frac{6}{7}x^{\frac{7}{6}} + 10x^{\frac{1}{2}}.$

5. $\frac{3}{8}x^{\frac{2}{3}}(12 + x^2).$

6. $\frac{2}{5\sqrt{x}}(x^3 + 5x^2 + 15x - 5).$

7. $\frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3.$

8. $\frac{1}{6}x^6 - \frac{3}{4}x^4 + \frac{3}{2}x^2 - \log x.$

9. $\frac{1}{3}x^3 - x^2 + 4x - 8\log(x+2).$

10. $\frac{2}{3}(1 + e^x)^{\frac{3}{2}}.$

11. $\log(e^x + a).$

12. $\log(\log x).$

13. $\log(\tan^{-1}x).$

14. $-\frac{1}{2(x + \sin x)^2}.$

15. $\log(e^x + e^{-x}).$

16. $\frac{2}{3b}\sqrt{(a + bx)^3}.$

17. $\log\frac{1}{\sqrt{3 - 4x + 4x^2}}.$

18. $-\frac{1}{9}(2 - 3x)^3.$

19. $-\frac{1}{b}\log(a + b\cos x).$

20. $-\frac{1}{4(1 + x^2)^2}.$

21. $-\frac{1}{2(2 + 3x^2)}.$

22. $\sqrt{5 + 4x + x^2}.$

23. $\frac{bx}{b'} + \frac{ab' - a'b}{b'^2}\log(a' + b'x).$

24. $\log\sqrt{e^{2x} + \tan 2x}.$

25. $\frac{1}{(1 - n)[\log(x + a)]^{n-1}}.$

26. $\frac{2}{3}(\tan^{-1}x - 1)^{\frac{3}{2}}.$

27. $\frac{1}{2}(\log x)^2.$

Page 34

28. $\frac{1}{6a}(e^{ax^2} + b)^3.$

29. $-2\sqrt{a^2 - x^2}.$

30. $x - \log(1 + e^x).$

31. $\log\left[\log\left(x + \sqrt{x^2 + a^2}\right)\right].$

32. $\frac{1}{3}[\log(x^2 + a^2)]^{\frac{3}{2}}.$

33. $\log\left(\sqrt[3]{x^5} - \sqrt[3]{x^4}\right)^3.$

34. $-\frac{\sqrt{ab}}{(\sqrt{ax} - \sqrt{bx})^2}.$

35. $\frac{1}{4}\sin^4 x.$

36. $\frac{1}{6a}[\sin^6(ax + b) - \cos^6(ax + b)].$

37. $\frac{1}{b}(\csc bx - \text{ctn}\, bx).$

38. $-\frac{1}{a}\text{ctn}\,(ax + b).$

39. $-\frac{1}{4}\text{ctn}^2(x^2 + a^2).$

40. $\log\sqrt{a^2 + \sec x^2}.$

41. $\frac{1}{3}\tan^3 x + \tan^2 x + \tan x.$

42. $\frac{1}{2}x - \frac{1}{8}\sin 4x.$

43. $\log(\csc x - \text{ctn}\, x) + 2\cos x.$

44. $-\frac{2}{3}\cos^3 x.$

45. $\tan x + \text{ctn}\, x.$

46. $\frac{2}{3}(\tan 3x - \sec 3x) - x.$

47. $2\tan x - \sec x - x.$

48. $\frac{1}{2}\left[\frac{\sin(m-n)x}{m-n}-\frac{\sin(m+n)x}{m+n}\right].$

49. $\frac{1}{2}\left[\frac{\sin(m+n)x}{m+n}+\frac{\sin(m-n)x}{m-n}\right].$

50. $-\frac{1}{2}\left\{\frac{\cos[(a+a')x+b+b']}{a+a'}+\frac{\cos[(a-a')x+b-b']}{a-a'}\right\}.$

51. $\frac{1}{2}(\tan 2x-\text{ctn}\,2x).$

52. $\frac{1}{4}(\sec x^2+\tan x^2)^2.$

53. $\frac{1}{2}x-\frac{1}{8}\sin(2-4x).$

54. $-2\cos x+\frac{1}{2}\log(\csc x-\text{ctn}\,x).$

55. $2x-\log(\sec 2x+\tan 2x).$

56. $\log(1-\cos x).$

57. $\sin\theta-\cos\theta.$

58. $\tan\theta-\sec\theta.$

59. $\sec\theta+\tan\theta-\theta.$

60. $2\tan\frac{x}{2}-x.$

61. $\csc x-\text{ctn}\,x.$

62. $\log(1-\csc x).$

63. $\frac{1}{15}\tan^{-1}\frac{3x}{5}.$

64. $\frac{1}{2}\sin^{-1}\frac{2x}{\sqrt{3}}.$

65. $\sec^{-1}5x.$

Page 35

66. $\frac{1}{\sqrt{3}}\sin^{-1}\sqrt{3}x.$

67. $\frac{1}{\sqrt{2}}\tan^{-1}\sqrt{2}x.$

68. $\frac{1}{\sqrt{3}}\sec^{-1}\frac{5x}{\sqrt{15}}.$

69. $\frac{1}{3}\tan^{-1}\frac{x-1}{3}.$

70. $\frac{1}{3}\sin^{-1}\frac{x^3}{a^3}.$

71. $\frac{1}{2a^2}\sec^{-1}\frac{x^2}{a^2}.$

72. $\frac{1}{2}\tan^{-1}\frac{x-3}{2}.$

73. $\sin^{-1}(x\cos\alpha-\sin\alpha).$

74. $\frac{1}{b}\sec^{-1}\frac{ax}{b}.$

75. $\frac{1}{\cos\alpha}\tan^{-1}(x\sec\alpha+\tan\alpha).$

76. $\sin^{-1}\frac{x-2}{2}.$

77. $\tan^{-1}(e^x+\tan\alpha).$

78. $\frac{1}{\sqrt{2}}\sec^{-1}\frac{x-2}{\sqrt{2}}.$

79. $\frac{1}{\sqrt{3}}\sin^{-1}(3x-1).$

80. $\sin^{-1}(e^x\sin\alpha+\cos\alpha).$

81. $\tan^{-1}(2x-1).$

82. $\frac{1}{3}\sec^{-1}\frac{2x+5}{3}.$

83. $\frac{1}{\sin^2\alpha}\tan^{-1}(x+\text{ctn}\,\alpha).$

84. $\frac{1}{2}\sin^{-1}\frac{x^2+3}{2\sqrt{3}}.$

85. $\sin^{-1}\frac{2x+3}{5}.$

86. $\frac{1}{5\sqrt{3}}\sec^{-1}\frac{3x-1}{5}.$

87. $\frac{1}{\sqrt{2}}\sin^{-1}\frac{4x+7}{9}.$

88. $2\tan^{-1}x+\frac{1}{2}x^2.$

89. $\frac{3}{4}\log(2x^2+1)+\sqrt{2}\tan^{-1}x\sqrt{2}.$

90. $\frac{1}{12}\log(3x^4+7)-\frac{\sqrt{21}}{6}\tan^{-1}\frac{3x^2}{\sqrt{21}}.$

91. $a\sin^{-1}\frac{x}{a}-\sqrt{a^2-x^2}.$

92. $2\sin^{-1}x.$

93. $-\tan^{-1}(\cos x).$

94. $\frac{1}{a\sqrt{a^2+b^2}}\tan^{-1}\frac{a\tan x}{\sqrt{a^2+b^2}}.$

95. $\frac{1}{12}\log\frac{2x-3}{2x+3}.$

96. $\frac{1}{3}\log\left(3x+\sqrt{9x^2+2}\right).$

97. $\frac{1}{2\sqrt{3}}\log\frac{3x-\sqrt{3}}{3x+\sqrt{3}}.$

98. $\frac{1}{2}\log\left(2x+\sqrt{4x^2-3}\right)$.

99. $\frac{1}{\sqrt{2}}\log\left(2x+\sqrt{4x^2-2}\right)$.

100. $\frac{1}{12\sqrt{2}}\log\frac{3x+\sqrt{2}}{3x-\sqrt{2}}$.

101. $\frac{1}{3}\log\left(x^3+\sqrt{x^6-a^6}\right)$.

102. $\frac{1}{4\sqrt{2}}\log\frac{x^2+2-\sqrt{2}}{x^2+2+\sqrt{2}}$.

103. $\log\left(x-\tan\alpha-\sqrt{x^2-2x\tan\alpha-1}\right)$.

Page 36

104. $\log\frac{2x+1}{x+1}$.

105. $\log\left(x+\sin\alpha+\sqrt{x^2+2x\sin\alpha+1}\right)$.

106. $\frac{1}{2\sqrt{3}}\log\frac{3x-3-\sqrt{3}}{3x-3+\sqrt{3}}$.

107. $\frac{1}{\sqrt{5}}\log\left(10x-3+2\sqrt{25x^2-15x-15}\right)$.

108. $\frac{1}{8}\log\frac{x+3}{x-5}$.

109. $\frac{1}{\sqrt{3}}\log\left(3x-1+\sqrt{9x^2-6x+3}\right)$.

110. $\log\frac{2x-1}{x-1}$.

111. $\frac{1}{\sqrt{2}}\log\left(2x+2+\sqrt{4x^2+8x-10}\right)$.

112. $\frac{3}{2}\sqrt{x^4-a^4}+\log\left(x^2+\sqrt{x^4-a^4}\right)$.

113. $\frac{1}{2}\log(2x^2+3x+1)+\frac{9}{2}\log\frac{2x+1}{x+1}$.

114. $\frac{2}{3}\log(9x^2-6x-3)+\frac{1}{2}\log\frac{x-1}{3x+1}$.

115. $2\sqrt{x^2+x+2}-4\log\left(2x+1+2\sqrt{x^2+x+2}\right)$.

116. $3\sqrt{x^2+4x-1}+\log\left(x+2+\sqrt{x^2+4x-1}\right)$.

117. $\frac{2}{3}\log(3x^2-4x+4)+\frac{\sqrt{2}}{6}\tan^{-1}\frac{3x-2}{2\sqrt{2}}$.

118. $\frac{5}{2}\log(x^2+4x+1)-\sqrt{3}\log\frac{x+2-\sqrt{3}}{x+2+\sqrt{3}}$.

119. $5\sin^{-1}\frac{x+2}{\sqrt{5}}+2\sqrt{1-4x-x^2}$.

120. $-\frac{3}{2}\sqrt{1+2x-2x^2}+\frac{13\sqrt{2}}{4}\sin^{-1}\frac{2x-1}{\sqrt{3}}$.

121. $\frac{1}{8}\log(4x^2+4x\sec\alpha+\tan^2\alpha)-\frac{\sec\alpha}{8}\log\frac{2x+\sec\alpha-1}{2x+\sec\alpha+1}$.

122. $-3\sqrt{1-2x-x^2}+5\sin^{-1}\frac{x+1}{\sqrt{2}}$.

123. $\frac{3}{4}\log(2x^2+6x+7)+\frac{1}{2\sqrt{5}}\tan^{-1}\frac{2x+3}{\sqrt{5}}$.

124. $\frac{1}{2}\log\left(x^2+\sqrt{x^4-a^4}\right)-\frac{1}{2}\sec^{-1}\frac{x^2}{a^2}$.

125. $\log\left(x+\sqrt{x^2-1}\right)+\sec^{-1}x$.

126. $\frac{a^x}{\log a}+\frac{x^{a+1}}{a+1}$.

127. $\frac{1}{3}(e^{3x}-e^{-3x})+3(e^x-e^{-x})$.

128. $\frac{1}{4\log a}(a^{2x^2}-a^{-2x^2})+x^2$.

129. $\frac{1}{2}e^{x^2+1}$.

130. $e^{-\frac{1}{x}}$.

131. $-\frac{a^{\cos 2x}}{4\log a}$.

132. $e^{\tan^{-1}x}$.

133. $\frac{e^{b+cx}a^{b+cx}}{c(1+\log a)}$.

134. $\frac{a^{2nx}}{2n\log a}+\frac{b^{2mx}}{2m\log b}+\frac{2a^{nx}b^{mx}}{n\log a+m\log b}$.

135. $\frac{1}{2}e^{\sin^{-1}x^2}$.

136. $\frac{1}{a^{\operatorname{ctn} x}e^{\csc x}}$.

137. $\frac{1}{\log a}\log\left(a^x+1+\sqrt{a^{2x}+2a^x+\sec^2\alpha}\right)$.

138. $\frac{1}{\log a}\sin^{-1}\frac{a^x-2}{2}$.

139. $\frac{1}{\log a}\tan^{-1}a^x$.

140. $2x-a\log\left(e^{\frac{x}{a}}+1\right)$.

141. $2\log(e^x+1)-x$.

Page 37

142. $-\frac{\sqrt{a^2-x^2}}{a^2x}$.

143. $\frac{x}{a^2\sqrt{a^2-x^2}}$.

144. $\sqrt{x^2-a^2}-a\sec^{-1}\frac{x}{a}$.

145. $\frac{x}{\sqrt{a^2-x^2}}-\sin^{-1}\frac{x}{a}$.

146. $\frac{\sqrt{x^2-a^2}}{2a^2x^2}+\frac{1}{2a^3}\sec^{-1}\frac{x}{a}$.

147. $\frac{3x^2-2a^2}{3(a^2-x^2)^{\frac{3}{2}}}$.

148. $\frac{1}{15}(3x^2-2a^2)(a^2+x^2)^{\frac{3}{2}}$.

149. $-\frac{x}{a^2\sqrt{x^2-a^2}}$.

150. $\frac{1}{a}\log\frac{\sqrt{a^2+x^2}-a}{x}$.

151. $\frac{1}{3}(x^2-2a^2)\sqrt{a^2+x^2}$.

152. $\frac{1}{ab}\sec^{-1}\frac{e^{ax}}{b}$.

153. $x+4\sqrt{x}+4\log(\sqrt{x}-1)$.

154. $\frac{1}{35}(5x^2-6x+6)(2x+3)^{\frac{3}{2}}$.

155. $\frac{1}{b^3}\log(a+bx)+\frac{3a^2+4abx}{2b^3(a+bx)^2}$.

156. $\frac{2}{15}(3x^2+4x+8)\sqrt{x-1}$.

157. $\frac{3}{28}(4x-3a)(x+a)^{\frac{4}{3}}$.

158. $\frac{1}{a}\sec^{-1}\frac{x}{a}$.

159. $\frac{1}{15}\sqrt{x^2+a^2}(3x^4-4a^2x^2+8a^4)$.

160. $\frac{1}{20}x^4-\frac{1}{50}\log(2+5x^4)$.

161. $\frac{14x^3-5}{49(5-7x^3)^2}$.

162. $-\frac{1+2x^2}{4(1+x^2)^2}$.

163. $\log\sqrt{x^2+4}+\frac{2}{x^2+4}$.

164. $\frac{1}{40}(4x^3-9)\sqrt[3]{(3+2x^3)^2}$.

165. $\cos^{-1}\dfrac{2x+1}{2\sqrt{2}\,x}$.

166. $\dfrac{1}{a}\sin^{-1}\dfrac{x-a}{x\sqrt{2}}$.

167. $\log\dfrac{x+2}{2x+5+\sqrt{x^2+8x+13}}$.

168. $\sin^{-1}\dfrac{x-2}{(x-1)\sqrt{3}}$.

169. $\sin^{-1}\dfrac{4x+11}{(x+3)\sqrt{15}}$.

170. $\frac{1}{5}\log\dfrac{x^5}{1+4x^5}$.

171. $-\dfrac{1}{9(1+3x^3)}$.

172. $-\dfrac{3\sqrt[3]{x+4x^6}}{5x^2}$.

173. $\dfrac{3x}{4\sqrt[3]{2x-x^3}}$.

174. $x(\log ax-1)$.

175. $x^{m+1}\left(\dfrac{\log ax}{m+1}-\dfrac{1}{(m+1)^2}\right)$.

176. $x\tan^{-1}ax-\dfrac{1}{a}\log\sqrt{1+a^2x^2}$.

177. $\log x\,[\log(\log x)-1]$.

Page 38

178. $x\log\left(x+\sqrt{x^2+a^2}\right)-\sqrt{x^2+a^2}$.

179. $\dfrac{1}{a^2}(\sin ax-ax\cos ax)$.

180. $x\sec^{-1}ax-\dfrac{1}{a}\log\left(ax+\sqrt{a^2x^2-1}\right)$.

181. $\dfrac{1}{2a^2}\left(a^2x^2\sec^{-1}ax-\sqrt{a^2x^2-1}\right)$.

182. $\dfrac{1}{2a^2}[(a^2x^2+1)\tan^{-1}ax-ax]$.

183. $\dfrac{e^{ax}}{a^3}(a^2x^2-2ax+2)$.

184. $e^x(x^2+1)$.

185. $\dfrac{x^2\sin ax}{a}+\dfrac{2x\cos ax}{a^2}-\dfrac{2\sin ax}{a^3}$.

186. $\dfrac{1}{a^4}[(3a^2x^2-6)\sin ax-(a^3x^3-6ax)\cos ax]$.

187. $x[(\log ax)^2-2\log ax+2]$.

188. $\frac{1}{27}x^3[9(\log ax)^2-6\log ax+2]$.

189. $\dfrac{1}{8a^2}[2a^2x^2-2ax\sin 2ax-\cos 2ax]$.

190. $\frac{1}{8}e^{2x}(2+\cos 2x+\sin 2x)$.

191. $\frac{1}{20}e^x[5(\cos x+\sin x)-(\cos 3x+3\sin 3x)]$.

192. $\frac{1}{2}e^{3x}[\frac{1}{25}(3\sin 4x-4\cos 4x)+\frac{1}{13}(3\sin 2x-2\cos 2x)]$.

193. $\dfrac{1}{4a^2}\left[(2a^2x^2-1)\sin^{-1}ax+ax\sqrt{1-a^2x^2}\right]$.

CHAPTER III

Page 61

1. $4\frac{3}{4}$.
2. $\frac{1}{4}\log\frac{5}{3}$.
3. $\frac{1}{15}a^3$.
4. $8a^3(\log 2-\frac{2}{3})$.
5. $2-\sqrt{2}$.
6. $\dfrac{1}{2}\left(1-\dfrac{1}{e}\right)$.
7. $\frac{1}{2}\log\frac{4}{3}$.
8. $\log(2+\sqrt{3})$.
9. $\frac{37}{24}$.
10. $2\log(e+1)-1$.
11. $\tan\sqrt{e}-\tan 1$.
12. $\dfrac{\pi}{2}$.
13. $\dfrac{\pi}{2}$.
14. $\frac{203}{480}$.
15. $\frac{8}{3}$.
16. $\frac{17}{480}$.
17. $\dfrac{\pi}{4a^3}$.
18. $\dfrac{1}{a^2\sqrt{2}}$.
19. $\frac{1}{2}\pi a^2$.

20. $\frac{1}{2}\left(x\sqrt{x^2-a^2}-a^2\log\frac{x+\sqrt{x^2-a^2}}{a}\right).$
21. $\pi.$
22. $\frac{8}{3}\log 2-\frac{7}{9}.$
23. $\frac{1}{12}\pi+\frac{1}{2}\sqrt{3}-1.$
24. $\frac{1}{18}(3\pi-4).$
25. $2-\frac{5}{e}.$

Page 62

35. $\frac{1}{4}\pi a.$
36. .6366.

Page 63

46. .4621.
47. .016.
48. .8746.

CHAPTER IV

Page 80

1. $170\frac{2}{3}.$
2. $\frac{1}{6}a^2.$
3. $4\pi a^2.$

Page 81

4. $a^2\left(e^{\frac{h}{a}}-e^{-\frac{h}{a}}\right).$
5. $\frac{4}{3}ab.$
6. $2\pi ab.$
7. $a^2\left(\frac{\pi}{4}-\frac{1}{2}\log 2\right).$
8. $a^2\cos^{-1}\frac{h}{a}-h\sqrt{a^2-h^2}.$
9. $\frac{8}{15}a^{\frac{5}{2}}.$
10. $4a^2.$
12. $\frac{1}{4}\pi a^2.$
13. $3\pi a^2.$
14. $2\pi+\frac{4}{3};\ 6\pi-\frac{4}{3}.$
15. $\frac{1}{3}.$
16. 32.
17. $a^2(2\pi-\frac{4}{3}).$
18. $2a^2.$
20. $\frac{4}{3}\pi^3a^2.$
21. $\frac{a^2}{\sqrt{3}}.$
22. $\frac{3}{2}\pi a^2.$
23. $\frac{1}{2}\pi(a^2+2b^2).$
24. $\frac{3}{4}\pi a^2.$
25. $\frac{a^2}{n}.$
26. $\frac{\pi a^2}{4n}.$
27. $a^2\left(2-\frac{\pi}{2}\right).$

Page 82

28. $\frac{1}{3}a^2\sqrt{2}.$
29. $2\pi(a^2+b^2).$
30. $\frac{1}{8}\pi a^2.$
31. $\frac{1}{15}\pi a^3.$
32. $\frac{32}{105}\pi a^3.$
33. $\frac{1}{4}\pi a^3\left(e^{\frac{2h}{a}}-e^{-\frac{2h}{a}}\right)+\pi a^2h.$
34. $4\pi a^3(2\log 2-1).$
35. $\frac{8}{3}\pi a^3(3\log 2-2).$
36. $\frac{4}{3}\pi(a^2-h^2)^{\frac{3}{2}}.$
37. $\pi^2a^2b+\frac{4}{3}\pi a^3;\ \pi^2a^2b-\frac{4}{3}\pi a^3$ (b is the distance from the diameter of semi-circle to the axis).
38. $\frac{4}{3}\pi ab^2.$
39. $2\pi^2abd.$
40. $\frac{\pi h^2b^2}{3a^2}(3a+h).$
41. $\frac{2\pi a^2h}{3b^2}(h^2+3b^2).$
42. $\frac{1}{4}\pi^2(2a^2+1)+2\pi a.$
43. $2ph^2.$
44. $\frac{\pi h^5}{80p^2}.$

Page 83

45. $\frac{\pi h}{120\,p^2}(3\,h^4 + 40\,pah^2 + 240\,p^2a^2)$.

46. $\frac{16}{15}\,\pi p^{\frac{1}{2}}h^{\frac{3}{2}}(5\,a - 3\,h)$.

47. $\frac{2}{3}\,\pi a^3(1 - \cos\alpha)$ (a is the radius of the sphere and $2\,\alpha$ is the vertical angle of the cone).

48. $6\,\pi(1 + 6\sqrt{6}\,\pi)$.

49. $\frac{8}{3}\,\pi a^3$.

50. $\frac{1}{3}\,\pi a^3$.

53. $\frac{4}{3}\,ab^2$.

54. $\frac{2}{3}\,a^3 \tan\theta$ (a is the radius of the cylinder).

55. $(\pi - \frac{4}{3})\,ha^2$ (a is the radius of the cylinder and h is the altitude).

56. $\frac{4}{3}\,ha^2$ (a is the radius of the cylinder and h is the altitude).

57. $2\,\pi h^2\sqrt{p_1p_2}$ (p_1 and p_2 are the respective parameters of the parabolas).

58. $\frac{4}{3}a^3\left(1 + \frac{\pi}{2}\right)$ (a is the radius of the sphere).

Page 84

59. $\frac{8}{3}\,\pi a^2b$.

60. $\frac{64}{105}\,a^3$ (a is the parameter of the hypocycloids).

61. $\frac{8}{3}\,ab^2$.

62. $\frac{16\,pa^3}{3\,q}$.

63. 1152 cu. in.

64. $\frac{1}{27}\,[(4+9\,h)^{\frac{3}{2}}-8]$.

65. $\log(e + e^{-1})$.

66. $6\,a$.

67. $\frac{a}{2}\left(e^{\frac{h}{a}} - e^{-\frac{h}{a}}\right)$.

68. $a\log\frac{a}{h}$.

69. $\frac{4\,(a^2 + ab + b^2)}{a + b}$.

70. $\frac{2}{3\sqrt{3\,p}}[(h+p)^{\frac{3}{2}}-(3\,p)^{\frac{3}{2}}]$.

71. $8\,a$.

72. $\frac{8\,a\,(a + b)}{b}$.

73. $\frac{1}{2}\,a\phi_1^2$.

74. 177.5 in.

75. 1007 ft.

Page 85

77. $\frac{8}{3}\,a$.

78. $\frac{3}{2}\,\pi a$.

79. $8\,a$.

80. $2\,\pi a\,(h_2 - h_1)$ (a is the radius of the sphere).

81. $\frac{8}{3}\,\pi\sqrt{p}\,[(h + p)^{\frac{3}{2}} - p^{\frac{3}{2}}]$.

82. $\frac{1}{2}\,\pi a^2\left(e^{\frac{2\,h}{a}} - e^{-\frac{2\,h}{a}}\right) + 2\,\pi ah$.

83. $2\,\pi a^2$.

84. $\frac{12}{5}\,\pi a^2$.

85. $\frac{64}{3}\,\pi a^2$.

86. $2\pi a^2 + \frac{2\,\pi ab^2}{\sqrt{a^2 - b^2}}\log\left(\frac{a + \sqrt{a^2 - b^2}}{b}\right)$.

87. $2\,\pi b^2 + \frac{2\,\pi ab}{e}\sin^{-1}e$ (e is the eccentricity of the ellipse).

88. $\frac{32}{5}\,\pi a^2$.

89. $4\,\pi a^2(2 - \sqrt{2})$.

CHAPTER V

Page 99

1. $\frac{m}{a}$.

2. $\frac{kx^2}{2\,a}$ (k is the proportionality factor).

3. $k\log\frac{v_2}{v_1}$; $\frac{k}{1-\gamma}(v_2^{1-\gamma} - v_1^{1-\gamma})$.

5. $\frac{k}{2\,\pi h}\log\frac{b}{a}$.

6. $\frac{M}{c\sqrt{c^2 + l^2}}$.

7. $\frac{M}{c\,(c + l)}$ (l is the length of the wire).

8. $\frac{2M}{lc}\sin\frac{\theta}{2}$ (l is the length of the wire) in a direction bisecting the angle θ.

9. $\frac{Mc}{(c^2+a^2)^{\frac{3}{2}}}$.

10. $\frac{2Mc}{a^2}\left(\frac{1}{c}-\frac{1}{\sqrt{c^2+a^2}}\right)$.

11. $\frac{2M}{a^2l}\left(l+\sqrt{c^2+a^2}-\sqrt{(c+l)^2+a^2}\right)$.

Page 100

12. $\frac{2M}{a^2\alpha}\sin\frac{\alpha}{2}$.

14. $\frac{1}{2}bw(a^2+2ac)$.

15. $\frac{2(a^2+3ac+3c^2)}{3(a+2c)}$ below the surface.

16. $\frac{1}{3}ba^2w$.

18. $\frac{1}{6}ba^2w$.

20. $\frac{1}{3}baw(2a+3c)$.

21. $\frac{4}{5}ba^2w$.

22. $\frac{5}{7}a$ below the surface.

23. $\frac{8}{15}ba^2w$.

24. $\frac{4}{7}a$ below the surface.

25. $\frac{2}{3}ab^2w$ (the axis $2a$ being in the surface).

26. $\frac{3}{16}\pi b$ below the surface.

27. 2250 lb.

28. 781.25 lb.

Page 101

29. $2234\frac{3}{8}$ ft.-lb.

30. 769.4 lb.

31. $1041\frac{2}{3}$ tons.

32. $8333\frac{1}{3}$ foot-tons.

33. 88.4 tons.

34. $4\frac{1}{5}$ tons

35. $\left(0, \frac{2a}{\pi}\right)$.

36. $(\frac{2}{5}a, \frac{2}{5}a)$.

37. $(0, \frac{2}{5}a)$.

38. $\left(\frac{7a}{5}, -\frac{a\sqrt{3}}{4}\right)$.

39. On the axis, three fifths of the distance from the vertex to the chord.

40. $(\frac{5}{7}k, 0)$ ($x=k$ is the equation of the ordinate).

41. $\left(\frac{4a}{3\pi}, \frac{4b}{3\pi}\right)$.

42. $\left(\frac{a}{5}, \frac{a}{5}\right)$.

43. $(\frac{4}{7}a, \frac{5}{32}a)$.

Page 102

45. $\left(\frac{1}{2}\pi, \frac{1}{8}\pi\right)$.

46. $\frac{1}{h\sqrt{\pi}}$.

47. Intersection of the medians.

48. $\left(\frac{8p}{5m^2}, \frac{2p}{m}\right)$.

49. $\left(\frac{9}{5}p, \frac{9}{5}p\right)$.

50. $\left(\frac{a}{2}, \frac{b}{2}+\frac{a^2}{12p}\right)$.

51. $\left(0, \frac{176p}{5(2+3\pi)}\right)$.

52. $\left(\frac{4a}{3\pi}, \frac{4(a+b)}{3\pi}\right)$.

53. On the radius perpendicular to the base of the hemisphere and three eighths of the distance from the base.

54. On the diameter of the sphere perpendicular to the planes, at a distance $\frac{1}{2}(h_1+h_2)$ from the center.

55. $\bar{x}=\frac{2}{3}a$.

56. $\bar{y}=\frac{5}{6}k$.

57. $\bar{x}=\frac{5}{16}b\ (b^2=4pa)$.

58. $\bar{y}=\frac{1}{2}y_1$ (y_1 is the ordinate of the point of intersection of the line and the parabola).

59. $\bar{y}=\frac{9}{16}b$.

60. At the middle point of the radius of the hemisphere perpendicular to the base.

61. On the axis of the cone, two thirds of the distance from the vertex to the base.

62. On the radius of the hemisphere perpendicular to the base, two thirds of the distance from the base to the vertex.

CHAPTER VI

Page 117

1. $\dfrac{2}{3x} - \dfrac{1}{3(2x+3)} - \dfrac{1}{3x-2}$.

2. $x + 1 + \dfrac{1}{2x} + \dfrac{1}{x-1} - \dfrac{1}{2(x+2)}$.

3. $\dfrac{7}{6(2x+1)} + \dfrac{1}{2(2x-1)} - \dfrac{1}{3(x+2)}$.

4. $1 - \dfrac{3}{2(x+1)} - \dfrac{1}{x-2} + \dfrac{1}{2(x+3)}$.

5. $\dfrac{1}{x} + \dfrac{2}{x^2} - \dfrac{3}{x^3} - \dfrac{1}{x+1}$.

6. $\dfrac{2}{x^2} + \dfrac{1}{x} - \dfrac{5}{(2x-1)^2} - \dfrac{2}{2x-1}$.

7. $\dfrac{2}{x+2} - \dfrac{3}{(x+2)^2} - \dfrac{2}{(x+2)^3} - \dfrac{1}{x}$.

8. $x - 2 - \dfrac{3}{(2x+1)^3} + \dfrac{1}{(2x+1)^2} + \dfrac{4}{2x+1}$.

9. $\dfrac{3}{2x+1} - \dfrac{x}{x^2+2}$.

10. $\dfrac{2}{x+2} + \dfrac{x-2}{x^2-2x+2}$.

11. $4x + \dfrac{3}{2(x-1)} + \dfrac{3}{2(x+1)} + \dfrac{2}{3x^2+1}$.

12. $\dfrac{2x}{x^2+x+2} + \dfrac{1}{x^2-x+2}$.

13. $x - 1 - \dfrac{2x}{x^2+1} + \dfrac{1}{x^2+2}$.

14. $\dfrac{2}{3x} + \dfrac{2}{(x^2+3)^2} + \dfrac{x}{3(x^2+3)}$.

15. $\dfrac{1}{x-2} + \dfrac{2x}{(x^2-2)^2} + \dfrac{1}{x^2-2}$.

16. $\dfrac{2x+5}{(x^2+x+1)^2} + \dfrac{x-2}{x^2+x+1}$.

17. $\log\left[(2x-3)^{\frac{3}{2}}(2x+5)^2\right]$.

18. $\log\sqrt[3]{\dfrac{3x+1}{(3x-2)^2}}$.

19. $\log\dfrac{(x+4)^2}{\sqrt{2x-3}}$.

20. $\dfrac{3}{2}\log(x^2+4x+1) + \dfrac{1}{2\sqrt{3}}\log\dfrac{x+2-\sqrt{3}}{x+2+\sqrt{3}}$.

21. $\frac{5}{8}\log(4x^2+4x+2) + \frac{3}{4}\tan^{-1}(2x+1)$.

22. $x - \frac{1}{2}x^2 - \log\left[(x-1)(x+3)^5\right]$.

23. $\dfrac{1}{2}x^2 - x + \dfrac{1}{6}\log(9x^2+12x+8) + \dfrac{1}{2}\tan^{-1}\dfrac{3x+2}{2}$.

24. $\dfrac{1}{2}x^2 + 4x + 6\log(x^2-2x-1) + 4\sqrt{2}\log\dfrac{x-1-\sqrt{2}}{x-1+\sqrt{2}}$.

25. $\log\dfrac{x(x+3)^3}{(x-2)^2}$.

26. $\log\dfrac{x^{15}(x+1)}{(x-1)^2}$.

27. $\log\dfrac{(x-1)(x+2)^2}{(x-3)^3}$.

28. $\log\dfrac{(x+1)(x+2)^4}{(x-2)^2}$.

29. $2x + \log\dfrac{(2x-1)^{\frac{1}{2}}}{x^{\frac{1}{3}}(x+3)^{\frac{14}{3}}}$.

30. $\frac{3}{2}x^2 - 2x + \frac{2}{3}\log(x+1) + \frac{1}{4}\frac{1}{2}\log(2x-1) + \frac{11}{7}\log(x+3)$.

31. $\frac{1}{2}(x^2+x) + \frac{1}{12}\log\frac{(2x-3)^2(2x+3)}{(x+2)^{12}}$.

32. $\log[x^3(2x-1)] + \frac{2}{2x-1}$.

33. $\log(x+2) + \frac{2x+2}{(x+2)^2}$.

34. $\log\left[(x-1)\sqrt{2x+3}\right] + \frac{3}{2(2x+3)}$.

Page 118

35. $-\frac{1}{2x+1} - \log(x+2)$.

36. $\log\sqrt{\frac{x-2}{x+2}} - \frac{3}{x+2}$.

37. $\log\frac{x}{(x+1)^3} - \frac{2}{x(x+1)}$.

38. $x + \frac{1}{3x} + \frac{2}{3}\log(3x-2)$.

39. $x - \frac{3}{2(x-1)^2} + \log x$.

40. $\frac{1}{2}(x^2-x) + \frac{4x+1}{8(4x^2-1)}$.

41. $\log\left[(x-1)\sqrt{(x^2+2x+2)^3}\right] + 4\tan^{-1}(x+1)$.

42. $\log\frac{(x+1)^2}{\sqrt{x^2-4x+2}} - \frac{3}{\sqrt{2}}\log\frac{x-2-\sqrt{2}}{x-2+\sqrt{2}}$.

43. $\frac{3}{2}\log(x^2+4) + \log\sqrt[4]{\frac{x-2}{x+2}} - \frac{1}{2}\tan^{-1}\frac{x}{2}$.

44. $\frac{1}{a+b}\log\sqrt{\frac{x^2-b}{x^2+a}}$.

45. $\frac{5}{2}\log\frac{x^2-2x+3}{3x^2+2} + \frac{1}{\sqrt{2}}\tan^{-1}\frac{x-1}{\sqrt{2}} - \frac{3}{\sqrt{6}}\tan^{-1}\frac{3x}{\sqrt{6}}$.

46. $\log\sqrt{\frac{x^2+3}{2x^2+x+5}} + \frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - \frac{5}{\sqrt{39}}\tan^{-1}\frac{4x+1}{\sqrt{39}}$.

47. $x - \frac{3}{2}\log(x^2-2x+2) - 3\tan^{-1}(x-1) - \tan^{-1}(x+1)$.

48. $\frac{1}{4}(x^2-2x) + \frac{1}{16}\log\frac{(2x+3)^4}{4x^2-6x+9} + \frac{13}{24\sqrt{3}}\tan^{-1}\frac{4x-3}{3\sqrt{3}}$.

49. $\frac{3}{2}\log(x^2-3) + \frac{1}{\sqrt{3}}\log\frac{x-\sqrt{3}}{x+\sqrt{3}} + \frac{10}{\sqrt{51}}\tan^{-1}\frac{6x+3}{\sqrt{51}}$.

50. $\frac{1}{2\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - \frac{2+x}{2(x^2+3)}$.

51. $\frac{1}{4}\log(2x^2+1) + \frac{5}{2\sqrt{2}}\tan^{-1}x\sqrt{2} - \frac{3+10x}{4(2x^2+1)}$.

52. $3\log\frac{\sqrt{x^2+3}}{x} + \frac{17}{3\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + \frac{2x+3}{3(x^2+3)}$.

53. $\log\frac{x^2}{2x^2-1} - \frac{13}{4\sqrt{2}}\log\frac{x\sqrt{2}-1}{x\sqrt{2}+1} - \frac{3+7x}{2(2x^2-1)}$.

CHAPTER VII

Page 137

1. $\frac{3}{2}\sqrt[3]{x^2} - \frac{3}{2}\log\left(\sqrt[3]{x^2}+1\right).$

2. $2\sqrt{x} + \log\dfrac{\sqrt{x}-1}{\sqrt{x}+1}.$

3. $\frac{2}{3}\left(2\sqrt[4]{x^3} - 3\sqrt{x} + 12\sqrt[4]{x}\right) - 8\log\left(\sqrt[4]{x}+1\right).$

4. $-\frac{6}{7}(x-2)^{\frac{7}{6}}.$

5. $-\frac{3}{4}(x+1)^{\frac{4}{3}} - (x+1) - \frac{3}{2}(x+1)^{\frac{2}{3}}.$

6. $2\sqrt{1-x} + \sqrt{2}\log\dfrac{\sqrt{1-x}-\sqrt{2}}{\sqrt{1-x}+\sqrt{2}}.$

7. $\sqrt[3]{1+3x} - 4\sqrt[6]{1+3x} + 4\log\left(\sqrt[6]{1+3x}+2\right).$

8. $1 + x + 4\sqrt{1+x} + 2\log\left(x-\sqrt{1+x}\right) + \dfrac{6}{\sqrt{5}}\log\dfrac{2\sqrt{1+x}-1-\sqrt{5}}{2\sqrt{1+x}-1+\sqrt{5}}.$

9. $x - 2\sqrt{x} + \log\left(\sqrt{x}+1\right)^2.$

10. $-\dfrac{(1+x^6)^{\frac{4}{3}}}{8x^8}.$

11. $\dfrac{x}{\sqrt[5]{1+2x^5}}.$

12. $-\dfrac{\sqrt[3]{3+x^3}}{3x}.$

13. $\dfrac{x^3+9}{2\sqrt[3]{x^3+3}}.$

14. $\frac{1}{440}(8x^5-1)(3x^5+1)^{\frac{8}{3}}.$

15. $\frac{1}{160}(5x^4-2)(2+3x^4)^{\frac{5}{3}}.$

16. $\dfrac{1}{3}\left[\dfrac{1}{2\sqrt{2}}\log\dfrac{\sqrt{3x^3+2}-\sqrt{2}}{\sqrt{3x^3+2}+\sqrt{2}} + \dfrac{1}{\sqrt{3x^3+2}}\right].$

17. $\frac{1}{4}x^2\sqrt{4+x^4} + \log\left(x^2+\sqrt{4+x^4}\right).$

18. $\sqrt{2}\tan^{-1}\dfrac{x+\sqrt{x^2+3x-2}}{\sqrt{2}}.$

19. $\dfrac{1}{\sqrt{5}}\log\dfrac{x\sqrt{2}+\sqrt{2x^2+3x+5}-\sqrt{5}}{x\sqrt{2}+\sqrt{2x^2+3x+5}+\sqrt{5}}.$

20. $\dfrac{1}{\sqrt{3}}\log\dfrac{\sqrt{3+3x}-\sqrt{3-x}}{\sqrt{3+3x}+\sqrt{3-x}}.$

21. $\dfrac{2(2x+1)}{3\sqrt{x^2+x+1}}.$

22. $\sqrt{x^2+2x+3} - \log\left(1+x+\sqrt{x^2+2x+3}\right).$

23. $-\dfrac{2(1+6x)}{3x}\sqrt{\dfrac{1-3x}{x}}.$

24. $\dfrac{2(28x^3+21x^2+12x-8)}{243(1-x-2x^2)^{\frac{3}{2}}}.$

25. $\dfrac{17x-6}{25\sqrt{2-3x-2x^2}} - \dfrac{1}{\sqrt{2}}\tan^{-1}\sqrt{\dfrac{4+2x}{1-2x}}.$

26. $\frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x.$

27. $\cos^{\frac{2}{3}}3x\left(\frac{1}{11}\cos^3 3x - \frac{1}{5}\cos 3x\right).$

Page 138

28. $\frac{1}{7}\cos^7 x - \frac{3}{5}\cos^5 x + \cos^3 x - \cos x.$

29. $\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x.$

30. $\frac{1}{6}\cos(2x+1)\left[\cos^2(2x+1) - 3\right].$

31. $x - \frac{1}{4}\cos 4x.$

32. $\left(\cos\dfrac{x}{3} + \sin\dfrac{x}{3}\right)\left(\sin\dfrac{2x}{3} - 5\right).$

33. $\dfrac{1}{a}\left(\dfrac{1}{4}\sin^4 ax - \dfrac{1}{6}\sin^6 ax\right).$

34. $\frac{1}{16}x - \frac{1}{64}\sin 4x - \frac{1}{48}\sin^3 2x.$

35. $\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x.$

36. $\frac{5}{16}x + \frac{1}{8}\sin 4x - \frac{1}{96}\sin^3 4x + \frac{3}{128}\sin 8x.$

37. $\frac{3}{128}x - \frac{1}{64}\sin 2x + \frac{1}{512}\sin 4x.$

38. $\frac{3}{14}\sqrt[3]{\cos 2x}\,(\cos^2 2x - 7).$

40. $\frac{1}{16}\log\frac{1+\cos 4x}{1-\cos 4x} - \frac{\cos 4x}{8\sin^2 4x}.$

39. $\log\frac{1+\sin\frac{x}{2}}{1-\sin\frac{x}{2}} - \frac{2}{3}\sin^3\frac{x}{2} - 2\sin\frac{x}{2}.$

41. $\frac{1}{8}\log\frac{1-\cos 2x}{1+\cos 2x} - \frac{\cos 2x}{4\sin^2 2x}.$

42. $\frac{\sin 3x}{12\cos^4 3x} + \frac{\sin 3x}{8\cos^2 3x} - \frac{1}{16}\log\frac{1-\sin 3x}{1+\sin 3x}.$

43. $\frac{1}{6}\tan^2 3x + \frac{1}{3}\log\cos 3x.$

44. $\frac{1}{3}\log\sin 3x + \frac{1}{6}\operatorname{ctn}^2 3x - \frac{1}{12}\operatorname{ctn}^4 3x.$

45. $\frac{2}{3}\tan^3\frac{x}{2} - 2\tan\frac{x}{2} + x.$

48. $2\tan\frac{x}{2}\left(1 + \frac{2}{3}\tan^2\frac{x}{2} + \frac{1}{5}\tan^4\frac{x}{2}\right).$

46. $-\frac{3}{5}\operatorname{ctn}^5\frac{x}{3} + \operatorname{ctn}^3\frac{x}{3} - 3\operatorname{ctn}\frac{x}{3} - x.$

47. $\frac{1}{2}(\tan^2 x - \operatorname{ctn}^2 x) + 2\log\tan x.$

49. $-\frac{4}{3}\operatorname{ctn}\frac{x}{4}\left(3 + \operatorname{ctn}^2\frac{x}{4}\right).$

50. $-\frac{1}{3}\operatorname{ctn} 3x\,(1 + \operatorname{ctn}^2 3x + \frac{3}{5}\operatorname{ctn}^4 3x + \frac{1}{7}\operatorname{ctn}^6 3x).$

51. $\frac{\sin 2x}{4\cos^2 2x} - \frac{1}{8}\log\frac{1-\sin 2x}{1+\sin 2x}.$

52. $-\frac{5\cos\frac{x}{5}}{4\sin^4\frac{x}{5}} - \frac{15\cos\frac{x}{5}}{8\sin^2\frac{x}{5}} + \frac{15}{16}\log\frac{1-\cos\frac{x}{5}}{1+\cos\frac{x}{5}}.$

53. $-\frac{1}{4}\operatorname{ctn}^4\frac{x}{3}\left(3 + 2\operatorname{ctn}^2\frac{x}{3}\right).$

54. $\frac{1}{a}\left(\sin ax + 2\csc ax - \frac{1}{3}\csc^3 ax\right).$

55. $\frac{2}{5}\sec^5\frac{x}{2}.$

56. $25\sqrt[5]{\sec\frac{x}{5}}\left(1 - \frac{2}{11}\sec^2\frac{x}{5} + \frac{1}{21}\sec^4\frac{x}{5}\right).$

57. $\tan^{\frac{7}{2}}x\,(\frac{2}{7} + \frac{4}{11}\tan^2 x + \frac{2}{15}\tan^4 x).$

58. $\frac{\sin 5x}{20\cos^4 5x} - \frac{\sin 5x}{40\cos^2 5x} + \frac{1}{80}\log\frac{1-\sin 5x}{1+\sin 5x}.$

59. $\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right).$

60. $\frac{1}{2}\tan^{-1}\left(\frac{1}{2}\tan\frac{x}{2}\right).$

62. $\frac{1}{\sqrt{13}}\log\frac{2\tan\frac{x}{2} + 3 - \sqrt{13}}{2\tan\frac{x}{2} + 3 + \sqrt{13}}.$

61. $\frac{1}{\sqrt{2}}\tan^{-1}\frac{3\tan\frac{x}{2} + 1}{2\sqrt{2}}.$

63. $\frac{1}{2\sqrt{7}}\log\frac{3\tan x - 4 - \sqrt{7}}{3\tan x - 4 + \sqrt{7}}.$

64. $\frac{1}{8}x\,(2x^2 + 5a^2)\,(x^2 + a^2)^{\frac{1}{2}} + \frac{3}{8}a^4\log\left(x + \sqrt{x^2 + a^2}\right).$

65. $\frac{1}{8}x\,(5a^2 - 2x^2)\,(a^2 - x^2)^{\frac{1}{2}} + \frac{3}{8}a^4\sin^{-1}\frac{x}{a}.$

66. $\frac{1}{2}\left[x\sqrt{x^2+a^2}-a^2\log\left(x+\sqrt{x^2+a^2}\right)\right].$

67. $-\frac{\sqrt{x^2+a^2}}{2\,a^2x^2}-\frac{1}{2\,a^3}\log\frac{x}{\sqrt{x^2+a^2}+a}.$

68. $-\frac{\sqrt{a^2-x^2}}{2\,a^2x^2}+\frac{1}{2\,a^3}\log\frac{x}{\sqrt{a^2-x^2}+a}.$

69. $\frac{1}{8}x(2\,x^2+a^2)\sqrt{x^2+a^2}-\frac{a^4}{8}\log\left(x+\sqrt{x^2+a^2}\right).$

70. $\frac{1}{8}x(2\,x^2-a^2)\sqrt{a^2-x^2}+\frac{a^4}{8}\sin^{-1}\frac{x}{a}.$

71. $-\frac{1}{15}(3\,x^2+2\,a^2)(a^2-x^2)^{\frac{3}{2}}.$

72. $-\frac{x}{2\,(a^2+x^2)}+\frac{1}{2\,a}\tan^{-1}\frac{x}{a}.$

73. $\frac{x}{(1+x^3)^{\frac{1}{3}}}.$

74. $\frac{2}{9}(x^3-2)(1+x^3)^{\frac{1}{2}}.$

75. $-\frac{(1+x^4)^{\frac{1}{4}}}{x}.$

76. $-\frac{\sqrt[3]{1+x^3}}{x}.$

77. $-\frac{\sqrt{2\,ax-x^2}}{ax}.$

78. $\sqrt{2\,ax-x^2}+a\sin^{-1}\frac{x-a}{a}.$

79. $\frac{1}{6}(2\,x^2-ax-3\,a^2)\sqrt{2\,ax-x^2}+\frac{a^3}{2}\sin^{-1}\frac{x-a}{a}.$

80. $-\frac{1}{2}(x+3\,a)\sqrt{2\,ax-x^2}+\frac{3\,a^2}{2}\sin^{-1}\frac{x-a}{a}.$

81. $-\frac{1}{6}\sin x\cos^5 x+\frac{1}{24}\sin x\cos^3 x+\frac{1}{16}\sin x\cos x+\frac{1}{16}x.$

82. $-\frac{1}{\sin x\cos^2 x}+\frac{3\sin x}{2\cos^2 x}+\frac{3}{2}\log(\sec x+\tan x).$

83. $-\frac{\cos x}{2\sin^2 x}+\frac{1}{2}\log(\csc x-\text{ctn}\,x).$

84. $\frac{1}{9}\tan^3 3\,x+\frac{1}{3}\tan 3\,x.$

85. $\frac{2\sin^2 x-1}{4\cos^4 x}.$

86. $\frac{2\cos^3 2\,x-3\cos 2\,x}{4\sin^2 2\,x}-\frac{3}{4}\log(\csc 2\,x-\text{ctn}\,2\,x).$

Page 139

87. $\frac{b}{a}\;h\sqrt{h^2-a^2}+a^2\log\frac{h-\sqrt{h^2-a^2}}{a}\Bigg).$

88. $\frac{8}{15}\cdot\frac{(b-a)^{\frac{5}{2}}}{c^{\frac{1}{2}}}.$

89. $\pi a^2.$

90. $\frac{ab}{30}.$

91. $\frac{1}{2}(\pi-2)\,a^2.$

92. $\frac{8}{15}.$

93. $\frac{3}{8}\pi a^2.$

94. $\frac{3}{8}\pi ab.$

95. $3\,\pi a^2.$

96. $\frac{1}{2}(4-\pi)\,a^2.$

97. $\frac{1}{2}(4+\pi)\,a^2.$

98. $\frac{\pi(a^2+b^2)}{4\,n}.$

99. $\frac{3}{4}\pi ab.$

100. $\frac{3}{2}a^2.$

101. $\frac{a}{2}\left[2\,\pi\sqrt{4\,\pi^2+1}+\log\left(2\,\pi+\sqrt{4\,\pi^2+1}\right)\right].$

102. $4\sqrt{2}\,a.$

103. $5\,\pi^2a^3.$

104. $4\,\pi^2a^3.$

105. $2\,\pi^2a^3.$

106. $\pi a^3\tan\theta.$

107. $\frac{1}{6}ha\left(3\,\pi b+8\sqrt{2}\,a\right).$

Page 140

108. $(\pi a,\ \frac{4}{3} a)$.

109. $(\pi a,\ \frac{5}{6} a)$.

110. $\left(\frac{a}{2},\ \frac{5a}{2(4\pi - 3\sqrt{3})}\right)$.

111. $\left(\frac{256a}{315\pi},\ \frac{256a}{315\pi}\right)$.

112. $\left(\frac{a}{3}(\sqrt{2}+1),\ 0\right)$.

CHAPTER VIII

Page 154

1. $3x^2 + 2x^3 - 3y^2 - 2y^3 = c$.

2. $e^{x^2} \sin y = c$.

3. $\sqrt{1+x^2} + \sqrt{1+y^2} = c$.

4. $\tan x = c \cos y$.

5. $x + y + \frac{1}{x} - \frac{1}{y} = c$.

6. $x = ce^{\frac{y}{x}}$.

7. $x^2 = 2cy + c^2$.

8. $2 \log x + \left(\sin^{-1} \frac{y}{x}\right)^2 = c$.

9. $\cos x \sin y = c$.

Page 155

10. $x \sin \frac{y}{x} = c$.

11. $x\left(e^{\frac{y}{x}} - e^{-\frac{y}{x}}\right) = c$.

12. $(x+y)^5 = cy^4 e^{\frac{3x}{y}}$.

13. $x^2 + 4xy - y^2 - 6x - 2y = c$.

14. $(y+3)(x+y+1) = c$.

15. $(x+y)^2 + 2y = c$.

16. $2y = \sin x - \cos x + ce^{-x}$.

17. $y = (x+1)^2(e^x + c)$.

18. $y = \frac{1}{2}(x+1)^3 + c(x+1)$.

19. $x = ce^{-\sin \frac{y}{x}}$.

20. $y = e^x\left(1 - \frac{1}{x}\right) + \frac{c}{x}$.

21. $\log xy + \frac{1}{2}(x^2 - y^2) = c$.

22. $x^2 = c^2 + 2cy$.

23. $y = 1 + x^2 + c\sqrt{1+x^2}$.

24. $y = 1 + c\left(x - \sqrt{1+x^2}\right)$.

25. $y = c \cos x - 2 \cos^2 x$.

26. $x^2 + 3xy + 2y^2 = cx$.

27. $y = \frac{1}{2}(x^2 - 1) + c\frac{x+1}{x-1}$.

28. $y^2 = -\frac{1}{4}(2x^2 + 2x + 3) + ce^{2x}$.

29. $y^2 = \frac{x(1+x^2)}{2+cx}$.

30. $y = \frac{e^{\tan^{-1} x}}{e^{\tan^{-1} x} + c}$.

31. $x^2(1 - y^2) = c(1 + x^2)$.

32. $x^2 \log x^2 + (x^3 + y^3)^{\frac{2}{3}} = cx^2$.

33. $y(e^x + 1) = e^x + x + c$.

34. $(\sec x + \tan x)(1 + y^{-3}) = x + c$.

35. $4y = ce^{2x} - (2x^2 + 2x + 1)$.

36. $y = (x + c)e^{ax}$.

37. $y = -\frac{\sin ax}{a^2} + c_1 x + c_2$.

38. $y = e^x(x - 2) + c_1 x + c_2$.

39. $y = \frac{1}{6} x^3 \log x - \frac{5}{36} x^3 + c_1 x + c_2$.

40. $y = \pm \frac{1}{2}\left(x\sqrt{c_1^2 - x^2} + c_1^2 \sin^{-1} \frac{x}{c_1}\right) + c_2$.

41. $y = c_1 x^2 + c_2$.

42. $y = \frac{c_1}{2} \log(x - a) - \frac{(x-a)^2}{4c_1} + c_2$.

43. $(y+1)^2 = c_1 x + c_2$.

44. $y + c_1 \log(y - c_1) + x = c_2$.

45. $y = -\frac{x^2}{4} - \frac{x}{3} + c_1 x^4 + c_2$.

46. $y = \frac{c_1(e^{c_1 x} - c_2)}{e^{c_1 x} + c_2}$.

47. $y = x + c_1 \log \frac{x - c_1}{x + c_1} + c_2$.

48. $y = \frac{4c_1^2 e^{c_1(x+c_2)}}{(1 - e^{c_1(x+c_2)})^2}$.

49. $y = \pm \frac{1}{2c_1}\left[c_1 x\sqrt{c_1^2 x^2 - 1} - \log\left(c_1 x + \sqrt{c_1^2 x^2 - 1}\right)\right] + c_2$.

50. $y = c_1 \sin[k(x - c_2)]$.

51. $y = c_1 \cosh[k(x + c_2)]$.

Page 156

52. $y(x-4)^2=9$.

53. $\tan\frac{y}{2}=e^{x-1}$.

54. $\tan\frac{y}{2}=ce^{\pm x\sqrt{2}}$.

55. $4y^3=9(kx+c)^2$.

56. $y=-1+e^{x-2}$.

57. $y=cx^n$.

58. $y=ax^2+c$.

59. $r^n=c\sin n\theta$.

60. $x=n\log y+c$.

62. $y=\frac{1}{k}\cosh[k(x-c)]$. ($k$ is the constant ratio.)

63. $x^2+y^2=cx$.

64. $y=a\cosh\frac{x-c}{a}$. (a is the constant length.)

65. $y=\pm\frac{k^2}{8}\sin^{-1}\frac{8x-k^2}{k^2}\pm\frac{1}{2}\sqrt{k^2x-4x^2}+c$. ($k$ is the constant ratio.)

66. $y=k\cosh\frac{x-c}{k}$. (k is the constant ratio.)

67. $EIy=-w\left(\frac{lx^2}{2}-\frac{x^3}{6}\right)$.

68. $EIy=-\frac{w}{2}\left(\frac{l^2x^2}{2}-\frac{lx^3}{3}+\frac{x^4}{12}\right)$.

69. $EIy=\frac{w}{2}\left(\frac{lx^2}{4}-\frac{x^3}{6}\right)$.

70. $(x-c_1)^2+(y-c_2)^2=c^2$. (c is the given constant.)

71. $c_1y^2-\frac{c_1^2}{k}(x+c_2)^2=1$. ($k$ is the constant ratio.)

Page 157

72. Harmonic motion.

73. $t=c+\sqrt{\frac{a}{2k}}\left(\sqrt{ax-x^2}-\frac{a}{2}\sin^{-1}\frac{2x-a}{a}\right)$, where k is the constant ratio, and $x=a$ when $v=0$.

74. Same velocity as if the body fell freely.

75. About 7 miles per second.

CHAPTER X

Page 195

1. $(1, -2, 2)$.

2. $(-\frac{19}{4}, -\frac{29}{2}, -\frac{19}{4})$.

3. $x^2+y^2+z^2-2x-4y-2z-43=0$.

6. $\frac{3}{2}+\frac{5}{8}\log 5$.

7. $e-\frac{1}{e}$.

8. $\pi\sqrt{2+4\pi^2}+\log\left(\pi\sqrt{2}+\sqrt{2\pi^2+1}\right)$.

11. $\cos^{-1}\frac{1}{\sqrt{2+t^2}}$.

13. $2x-3y+6z\pm 21=0$.

Page 196

14. $x+y+z-6=0$.

16. $ay+kz=0$.

17. $\pi x-2y-2z+2\pi=0$.

18. $e^t(x-e^t)-e^{-t}(y-e^{-t})+\sqrt{2}\left(z-t\sqrt{2}\right)=0$.

19. $x+2y+3z-6=0$.

20. $-\frac{9}{\sqrt{139}}, \frac{7}{\sqrt{139}}, \frac{3}{\sqrt{139}}$.

25. $x-2y+z+6=0$.

26. $\sin^{-1}\frac{13}{14}$.

27. $(1, 2, 1), (\frac{1}{3}, \frac{4}{3}, \frac{1}{3})$.

28. $x-z+2=0$.

29. $11x+13y-37=0, 3x+13z+10=0$.

Page 197

30. $(2, -1, 2)$.

31. $x-1=\frac{y-1}{3}=\frac{z-1}{4}$.

32. $\frac{x-1}{2}=\frac{y-2}{2}=z-1$.

33. $\frac{x-e^t}{e^t}=\frac{y-e^{-t}}{-e^{-t}}=\frac{z-t\sqrt{2}}{\sqrt{2}}$.

34. $x-z=0,\ y=0$.

35. $2y+z=0$.

36. $\cos^{-1}\frac{2}{\sqrt{13}},\ \frac{\pi}{2},\ \cos^{-1}\frac{3}{\sqrt{13}}$.

37. $(1, -1, 1)$.

39. $x+16y+7z+8=0$.

40. $3x-2y-z+4=0$.

41. $x+11y+5z=0$.

42. $x-6y+z-2=0$,
$3x+y+3z-1=0$.

43. $x-z=0$.

44. $93x-46y+13z-179=0$.

46. $(1, -2, 4),\ (\frac{8}{3}, \frac{14}{3}, \frac{17}{3})$.

47. $(0, 1, 2),\ (\frac{320}{37}, \frac{101}{37}, -\frac{214}{37})$.

CHAPTER XI

Page 218

10. $\Delta A = .396$ sq. in., $dA = .398$ sq. in.

11. $\Delta L = .057$ in., $dL = .057$ in.

12. $\Delta V = 5.11$ cu. ft., $dV = 5.09$ cu. ft.

13. $\frac{\sin\beta}{\sin(\alpha+\beta)}dh-\frac{h\sin\beta\cos(\alpha+\beta)}{\sin^2(\alpha+\beta)}d\alpha+\frac{h\sin\alpha}{\sin^2(\alpha+\beta)}d\beta$.

18. $x_1x+y_1y-(z_1-a)(z-a)=0$.

21. $\left(-\frac{ad}{a^2+b^2+c^2},\ -\frac{bd}{a^2+b^2+c^2},\ -\frac{cd}{a^2+b^2+c^2}\right)$.

22. $(a, a, a),\ (-a, -a, a),\ (-a, a, -a),\ (a, -a, -a)$.

23. Point of intersection of the medians.

24. $V=\frac{8abc}{3\sqrt{3}}$.

25. $x=\frac{2aK}{a^2+b^2+c^2},\ y=\frac{2bK}{a^2+b^2+c^2},\ z=\frac{2cK}{a^2+b^2+c^2}$.

Page 219

32. $\frac{x_1x}{a^2}+\frac{y_1y}{b^2}+\frac{z_1z}{c^2}=1$.

38. 0°.

39. $\cos^{-1}\frac{a^2-b^2+c^2}{2ac}$.

Page 220

41. $\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial x}+\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial y}+\frac{\partial f}{\partial z}\frac{\partial \phi}{\partial z}=0$.

44. $\frac{x-x_1}{\frac{ny_1}{b^2}-\frac{mz_1}{c^2}}=\frac{y-y_1}{\frac{lz_1}{c^2}-\frac{nx_1}{a^2}}=\frac{z-z_1}{\frac{mx_1}{a^2}-\frac{ly_1}{b^2}}$.

45. $\frac{x-x_1}{y_1z_1}=\frac{y-y_1}{-z_1x_1}=\frac{z-z_1}{x_1y_1}$.

46.
$$\left(-\frac{a^2cln}{\sqrt{(a^2l^2+b^2m^2)(a^2l^2+b^2m^2+c^2n^2)}},\right.$$
$$\left.-\frac{b^2cmn}{\sqrt{(a^2l^2+b^2m^2)(a^2l^2+b^2m^2+c^2n^2)}},\ \frac{c\sqrt{a^2l^2+b^2m^2}}{\sqrt{a^2l^2+b^2m^2+c^2n^2}}\right).$$

47. $(-1, -1, 1)$.

48. $\sin^{-1}\frac{k\sqrt{r^2-a^2}}{r\sqrt{a^2+k^2}}$.

49. 0°.

61. $xy\left(\frac{\partial^2V}{\partial y^2}-\frac{\partial^2V}{\partial x^2}\right)+(x^2-y^2)\frac{\partial^2V}{\partial x\partial y}-y\frac{\partial V}{\partial x}+x\frac{\partial V}{\partial y}$.

Page 221

68. $\frac{\partial^2 f}{\partial x^2}+2\frac{\partial^2 f}{\partial y \partial x}F'(x)+\frac{\partial^2 f}{\partial y^2}[F'(x)]^2+\frac{\partial f}{\partial y}F''(x).$

70. $\pi \log \frac{1+\sqrt{1-a^2}}{2}.$

71. $\log(a+1).$

CHAPTER XII

Page 234

1. $4\log 2-1\frac{1}{2}.$
2. $2.$
3. $\frac{3}{8}\pi^2+\frac{1}{2}\pi+1.$
4. $3(\log 2-1).$
5. $\frac{1}{8}\pi a^4.$
6. $6.$
7. $\frac{1}{4}\pi a^2.$
8. $\frac{1}{e}-\frac{1}{2}.$

Page 235

9. $\frac{1}{8}\pi^2 a^2.$
10. $\log\sqrt{2}.$
11. $\frac{1}{12}a^8.$
12. $\frac{1}{12}.$
13. $\frac{8}{15}\pi a^5.$
14. $\frac{1}{8}a^2.$

CHAPTER XIII

Page 253

1. $\frac{1}{2}.$
2. $\frac{8}{35}\sqrt{ab}(a+b)(5a^2+2ab+5b^2).$
3. $\frac{64a^4\sqrt{a}}{3465b^3}(11b^2+a).$
4. $\frac{16}{27}a^4\sqrt{3}.$
5. $\frac{1}{32}\pi a^4.$
6. $\frac{35}{16}\pi a^4.$
7. $\frac{1}{4}\pi a^4.$
8. $\frac{1}{4}\pi a^4.$
9. $\frac{1}{4}\pi a^4.$
10. $\frac{3}{2}\pi a^4.$
11. $\frac{5}{16}\pi a^4.$
12. $\frac{45}{64}a^4.$
13. $\frac{64a^5\sqrt{a}}{3465b^3}.$
14. $\frac{1}{24}(8+3\pi)a^4.$

Page 254

15. $\frac{21}{64}\pi a^4.$
16. $(10\pi-\frac{64}{3})a^4.$
17. $(\frac{15}{8}-\log 4)a^2.$
18. $(2\pi-\frac{4}{3})a^2.$
19. $10\pi.$
20. $\frac{8}{3}(a+b)\sqrt{ab}.$
21. $\frac{1}{8}(\pi-2)a^2.$
22. $\frac{1}{12}a^2(3\pi-8);$
    $\frac{1}{12}a^2(3\pi-8);$
    $\frac{1}{6}a^2(3\pi+8).$
23. $\frac{\sqrt{3}}{4}a^2.$
24. $8a^2.$
25. $2a^2(\pi-2).$
26. $2\sqrt{2}\pi a^2.$
27. $8a^2.$
28. $2a^2\csc 2\alpha.$
29. $x^2+(y+z\tan\alpha)^2=a^2;\ 2a^2(\operatorname{ctn}\alpha+\alpha\csc^2\alpha).$
30. $\frac{1}{9}(20-3\pi).$
31. $\frac{2}{3}a^2(\pi-2).$
32. $\frac{2}{9}a^2(20-3\pi).$

Page 255

33. $16a^2[\pi-\sqrt{2}-\log(1+\sqrt{2})].$
34. $(\frac{2}{5}a,\ \frac{2}{5}a).$
35. $\left(0,\ \frac{3a(5\pi+8)}{5(6\pi-4)}\right).$
36. $(\frac{5}{3}a,\ 0).$
37. $\left(\frac{32a}{15\pi},\ \frac{16a}{35\pi}\right).$
38. $\left(\frac{288a}{175\pi},\ \frac{288a}{175\pi}\right).$
39. $\left(\frac{256a}{315\pi},\ \frac{256a}{315\pi}\right).$
40. On the axis of the sector, $\frac{3a\sin\alpha}{4\alpha}$ distant from the center of the circle.
41. $\frac{128\sqrt{2}a}{105\pi}.$
42. $(\frac{5}{6}a,\ 0).$
43. $(\frac{1}{4}\pi a,\ 0).$
44. $(\frac{3}{8}a,\ \frac{3}{8}b,\ \frac{3}{8}c).$

45. $\left(\frac{5a}{4}, 0, \frac{5a(k_1+k_2)}{8}\right)$.

46. $\left(\frac{8a}{15}, \frac{16b}{15\pi}, \frac{16c}{15\pi}\right)$.

47. $\left(\frac{4a}{3\pi}, \frac{2b}{3}, \frac{8a}{9\pi}\right)$.

48. On the axis of the cone, $\frac{37a}{28}$ distant from its vertex.

49. On the axis of the cone, $\frac{5}{6}$ of distance from vertex to base.

Page 256

50. $(\frac{2}{5}a, \frac{2}{5}a, \frac{2}{5}a)$, the center of the sphere being at the origin, and the octant being in the first octant bounded by the coördinate planes.

51. On the axis of the cone, midway between the vertex and the base.

52. $\frac{1}{90}a^3$.
53. $\pi a^3(k_2 - k_1)$.
54. $\frac{8}{5}\pi abc$.
55. $5\frac{1}{3}$.
56. $\frac{1}{8}(\pi + 2)a^4$.
57. $\frac{1}{2}\pi a^2 b$.
58. $(\frac{17}{12} - \log 4)a^3$.
59. $3\frac{1}{5}$.
60. 9π.
61. $\frac{1}{4}\pi c^4(5a + b)$.
62. $\frac{3\pi a^4}{32b}$.
63. $\frac{4}{35}\pi a^3$.
64. $\frac{2}{9}a^3(3\pi - 4)$.
65. $\frac{11}{96}\pi a$.
66. $\frac{3}{32}\pi ab^4$.
67. $\frac{9}{2}a^3$.
68. $\frac{7}{12}\pi a^3$.
69. $\frac{1}{360}a^3$.
70. $\frac{2}{27}a^3(3\pi - 4)$.

Page 257

71. $\frac{32}{9}a^3$.
72. $\frac{5}{8}\pi^2 a^3$.
73. $\frac{4}{15}\pi\rho abc(b^2 + c^2)$.
74. $\frac{1}{2}\pi\rho a^4 h$.
75. $\frac{1}{60}k\pi h^6 \tan^4\alpha$. ($k$ is the constant ratio.)
76. $\frac{1}{6}k\pi a^6$. (k is the coefficient of variation.)
77. $\frac{4}{15}\pi\rho(r_2^5 - r_1^5)$.
78. $\frac{8}{9}ka^6$. (k is the coefficient of variation.)
79. $\frac{6M(1 - \cos\alpha)}{h^2 \tan^2\alpha}$.
80. $\frac{\pi\rho a(2b - a)}{b}$.
81. $\frac{4M}{3a^2}$. (M is the mass of the hemisphere.)
82. $\frac{4M(7 - 4\sqrt{2})}{15a^2}$. ($M$ is the mass of the hemisphere.)
83. $\frac{M\sqrt{3}}{12a^2}$. (M is the mass of the ring.)

CHAPTER XIV

Page 276

1. $\frac{3}{8}\pi a^2$.
2. $\frac{\pi a^2}{b}(3b - 2a)$.
3. $\frac{\pi a^2}{b}(2a + 3b)$.

Page 277

5. 2.
6. 0.
7. $\frac{k}{r}$. (k is the constant ratio, r is the distance.)
8. $-k\log r$ (k and r as in Ex. 7).
10. $\frac{1}{2}$; $\frac{5}{6}$; $\frac{1}{6}$; 0.
11. $\frac{5}{6}$; $\frac{3}{2}$; $-\frac{1}{2}$.
12. $\frac{10}{3}$; $\frac{17}{5}$.
13. -10; $-\frac{65}{6}$; 0.
14. $x^2 + y^2 - xy + x - y = c$.
15. $1 + y^2 + x^2y^2 = cx^2$.
16. $(x + y)^3 + 2y^3 = c$.
17. $x^2 = 2cy + c^2$.
18. $\log(x^2 + y^2) - \tan^{-1}\frac{x}{y} = c$.

Page 278

19. $x^3 + 3xy^2 = cy^2.$

20. $\tan^{-1}x + xy = c$

21. $e^{-\frac{x}{y}} + \log y = c.$

22. $\log x + \dfrac{y^3}{3x^3} = c.$

23. $xy + \log\dfrac{y}{x} = c.$

24. $x^5 - x^3y + x^2y^2 = c.$

25. $2x + \sin 2x + 4y\cos x = c.$

26. $\log\cos^2(x+y) + y^2 = c.$

27. $\log x - \dfrac{2}{xy} - \dfrac{1}{2x^2y^2} = c.$

28. $12y(x+1) - 4x^3 - 3x^4 = c.$

CHAPTER XV

Page 302

14. $-1 < x < 1.$

15. $-1 < x < 1.$

16. $-1 < x < 1.$

17. $-1 < x < 1.$

18. All values of x.

19. All values of x.

20. $-1 < x < 1.$

21. $-1 < x < 1.$

22. $-\dfrac{a}{b} < x < \dfrac{a}{b}.$

23. $1 + \dfrac{x^2}{|\underline{2}} + \dfrac{5x^4}{|\underline{4}} + \dfrac{61x^6}{|\underline{6}} + \cdots.$

24. $1 + x + x^2 + \dfrac{2x^3}{3} + \cdots.$

25. $x - \dfrac{x^2}{2} + \dfrac{x^3}{6} - \dfrac{x^4}{12} + \cdots.$

26. $1 + x + \dfrac{x^2}{2} - \dfrac{x^4}{8} - \cdots.$

27. $1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \cdots.$

28. $-\dfrac{x^2}{2} - \dfrac{x^4}{12} - \dfrac{x^6}{45} - \dfrac{17x^8}{2520} - \cdots.$

29. $\dfrac{\pi^2}{3} - 4\left(\dfrac{\cos x}{1^2} - \dfrac{\cos 2x}{2^2} + \dfrac{\cos 3x}{3^2} - \cdots\right).$

30.
$$\frac{\sinh a\pi}{a\pi} - \frac{2a\sinh a\pi}{\pi}\left(\frac{\cos x}{1^2+a^2} - \frac{\cos 2x}{2^2+a^2} + \cdots\right)$$
$$+ \frac{2\sinh a\pi}{\pi}\left(\frac{\sin x}{1^2+a^2} - \frac{2\sin 2x}{2^2+a^2} + \frac{3\sin 3x}{3^2+a^2} - \cdots\right).$$

31. $4\left(\dfrac{\sin x}{1} + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5} + \cdots\right).$

32. $\dfrac{\pi}{4} - \dfrac{2}{\pi}\left(\dfrac{\cos x}{1^2} + \dfrac{\cos 3x}{3^2} + \dfrac{\cos 5x}{5^2} + \cdots\right) - \left(\dfrac{\sin x}{1} - \dfrac{\sin 2x}{2} + \dfrac{\sin 3x}{3} - \cdots\right).$

33.
$$\frac{3\pi}{4} - \frac{2}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$$
$$+ \left(\frac{3\sin x}{1} - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right).$$

34.
$$\frac{\pi^2}{6} - 2\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right]$$
$$+ \frac{1}{\pi}\left[\left(\frac{\pi^2}{1} - \frac{4}{1^3}\right)\sin x - \frac{\pi^2}{2}\sin 2x + \left(\frac{\pi^2}{3} - \frac{4}{3^3}\right)\sin 3x - \frac{\pi^2}{4}\sin 4x + \cdots\right].$$

Page 303

35. $-\sin a.$

36. 1.

37. $-2.$

38. $\log\dfrac{a}{b}.$

39. 0.

40. $-2.$

41. $-\frac{1}{6}.$

42. 2.

43. 5.

44. 0.

45. 0.

46. 0.

47. 0.

48. 0.

49. $\frac{a_0}{b_0}$.
50. 2.
51. $\frac{7}{5}$.
52. 0.
53. ∞.
54. -1.
55. $\frac{1}{2}$.
56. -1.
57. 1.
58. 1.
59. 1.
60. 1.
61. 1.
62. $\frac{1}{e}$.
63. 1.
64. e.
65. e^{ab}.
66. 1.

CHAPTER XVII

Page 336

1. $y = a_0 + a_0^2 x + (a_0^3 - \frac{1}{2})x^2 + \left(a_0^4 - \frac{a_0}{3}\right)x^3 + (a_0^5 - \frac{5}{12}a_0^2)x^4 + \cdots$.
2. $y = a_0 + x + \frac{1}{3a_0}x^3 - \frac{1}{4a_0^2}x^4 + \frac{1}{5a_0^3}x^5 - \cdots$.
3. $y = a_0 + a_0^2 x + a_0^3 x^2 + a_0^4 x^3 + (a_0^5 + \frac{1}{4})x^4 + \cdots$.
4. $(y - 5x + c)(y + 2x + c) = 0$.
5. $(y - cx)(y^2 - x^2 - c^2) = 0$.
6. $(x^2 y - c)(y - cx) = 0$.
7. $(y - c)(y - ce^{-x^2})(3y - x^3 - c) = 0$.
8. $[x - a - c(x + a)e^y][x - a - c(x + a)e^{-y}] = 0$.
9. $y^2 \sin^2 x + 2cy + c^2 = 0$.
10. $cy^2 - 2c^2 xy + c^3 x^2 = 1$.
11. $y^2 = 2cx - c^2$.
12. $y^2 = 2cx + c^2$.
13. $c^2(1 + y^2) - 2c^3 xy + c^4 x^2 = 1$.
14. $x = \log p + p + c,\ y = p + \frac{1}{2}p^2$.
15. $y = c(c - x)^2$.

Page 337

16. $x^2 = \sin^2(y + c)$.
17. $y = p \log\sqrt{cp},\ x = \left(1 + \log\sqrt{cp}\right)\log\sqrt{cp}$.
18. $y = cx^2 + c^2$.
19. $x + y = \operatorname{ctn}(c - y)$.
20. $9(x + 2y + c)^2 = 4(x + 1)^3$.
21. $(y - x\log cx)(2x^2y^2 - x^4 - c) = 0$.
22. $x = \frac{c - p + \log p}{(p - 1)^2},\ y = \frac{p^2 c - p^3 + p^2\log p}{(p - 1)^2} + p$.
23. $\left(1 \pm \sqrt{y + 1}\right)^2 = ce^{-x \pm 2\sqrt{y+1}}$.
24. $\left(cy - e^{\frac{x^2}{2}}\right)(x^3 - 3y + c)(xy + cy + 1) = 0$.
25. $y = cp^2 e^{2p},\ x = \frac{c}{2}(1 + 2p)e^{2p}$.
26. $(1 - 4y)^2 + 4x^2 - 2c[(1 - 4y)\cos 2x + 2x\sin 2x] + c^2 = 0$.
27. $(y^2 - cx)^2 - c^2 + 1 = 0$.
28. $e^y = ce^x + c^3$.
29. $y^2 = cx^2 + \frac{1}{c}$.
30. $y = cx \pm \sqrt{1 - c^2},\ y^2 - x^2 = 1$.
31. $(2y - 3x^2 + c)(2y - x^2 + c) = 0,\ x = 0$.
32. $x^2 = 2cy + c^2 + a^2,\ x^2 + y^2 = a^2$.
33. $y = \frac{c}{x} + c^2,\ 1 + 4x^2 y = 0$.
34. $(y + cx)^2 = c,\ 4xy - 1 = 0$.
35. $(x - c)^2 + y^2 = a^2,\ y^2 = a^2$.
36. $c^2 + cxy + a^3 x = 0,\ xy^2 - 4a^3 = 0$.
37. $27y - 4x^3 = 0$.
38. $b^2x^2 + a^2y^2 - a^2b^2 = 0,\ y = cx + \sqrt{b^2 + a^2c^2}$.
39. $(x^2 + 1)(y^2 - 1)z^2 = c$.
40. $(x + y)(y + z)(z + x) = c$.
41. $x^3 + y^2 + z^2 + \log yz^2 = c$.
42. $xy + yz + zx - (b + c)x - (c + a)y - (a + b)z = k$.

43. $y(x+z)+c(y+z)=0.$

44. $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}=c.$

45. $\log(x+y)+x+y+z=c.$

46. $\frac{x}{y}+\frac{y}{z}-\log z=c.$

47. $\log z=\tan^{-1}\frac{y}{x}+c.$

48. $x^2+a^2=k_1(y^2+b^2),\ \frac{1}{b}\tan^{-1}\frac{y}{b}-\frac{1}{c}\tan^{-1}\frac{z}{c}=k_2.$

49. $x^2-y^2=c_1,\ y^2-z^2=c_2.$

50. $x=z+c_1,\ y^2=2xz+c_2.$

51. $e^{-\frac{1}{x}}+e^{-y^2}=c_1,\ e^{-\frac{1}{x}}-e^{-\frac{1}{z}}=c_2.$

52. $y=(x+c_1)\sqrt{x^2+1},\ z=(c_1\tan^{-1}x+c_2)\sqrt{x^2+1}.$

Page 338

53. $x^2+z^2=c_1^2,\ y+c_2=\frac{c_1}{2}\log\frac{z-c_1}{z+c_1}.$

54. $x^2-xy=c_1,\ z+x^2-c_2x+c_1=0.$

55. $x+y=c_1e^z,\ 2y=z^2+c_2.$

56. $x-y=c_1e^{\frac{z^2}{2}},\ y=c_2e^{-\frac{z^2}{2}}.$

57. $x-y=c_1z,\ x^2-y^2=c_2y.$

58. $x+y+z=c_1,\ x^2+y^2+z^2=c_2^2.$

59. $x+y+2z=c_1,\ x-y=c_2z^2.$

60. $y=c_1z,\ x^2+y^2+z^2=c_2z.$

61. $16y^3+27x^4=0.$

62. $4xy=c^2.$

63. $x^2-4y^2=0.$

64. $x^{\frac{1}{2}}+y^{\frac{1}{2}}=c^{\frac{1}{2}}.$

65. $2xy=\pm c^2.$

66. $27py^2=4(x-2p)^3.$

67. $3x^2-4xy=0.$

68. $y^2=4px+4p^2.$

69. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}}.$

70. $x^3+(x+2p)y^2=0.$

71. $(x-y)^2-2k(x+y)+k^2=0.$

72. $x^2-4a(a-y)=0.$

73. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}.$

74. $2(x-c)=k\log\left(k\pm\sqrt{k^2-4y^2}\right)\mp\sqrt{k^2-4y^2}.$

Page 339

75. $r=k.$

76. $r=ce^{\pm\frac{\sqrt{1-k^2}}{k}\theta}.$

77. $2x^2+y^2=c^2.$

78. $x^2+y^2=2a^2\log x+c.$

79. $y=cx^4.$

80. $y^2=4ax+4a^2.$

81. A family of circles tangent to OX at O.

82. $x^2+y^2-cx+1=0.$

84. A family of lemniscates having the line $\theta=\frac{\pi}{2}$ for a common axis.

85. $r=c(1-\cos\theta).$

86. $r=e^{\sqrt{c^2-\theta^2}}.$

CHAPTER XVIII

Page 360

1. $y=ce^{-\frac{2x}{3}}.$

2. $y=ce^{-3x}+\frac{1}{3}x^2-\frac{2}{9}x+\frac{2}{27}+\frac{1}{20}(3\sin x-\cos x).$

3. $y=ce^{2x}+e^{3x}+\frac{1}{2}e^x(\sin x-\cos x).$

4. $y=(c+3x)e^{-x}+\frac{1}{4}(2x-1)e^x.$

5. $y=ce^{-4x}+\frac{3}{4}-\frac{3}{10}(2\cos 2x+\sin 2x).$

6. $y=e^x[c+2x-\log(e^{2x}+1)].$

7. $y=ce^{2x}-\frac{1}{58}(2\sin 5x+5\cos 5x)+\frac{1}{10}(2\sin x+\cos x).$

8. $y = c_1e^{2x} + c_2e^{-4x}$.

9. $y = c_1e^{5x} + c_2$.

10. $y = (c_1 + c_2x)\,e^{-4x}$.

11. $y = c_1 \sin 3x + c_2 \cos 3x$.

12. $y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$.

13. $y = c_1e^{x} + c_2e^{-3x} - \frac{1}{3}x^4 - \frac{2}{9}x^3 - \frac{16}{9}x^2 - \frac{121}{27}x - \frac{338}{81}$.

14. $y = c_1e^{x} + c_2e^{-x} - 2 + \frac{2}{5}\cos 2x$.

15. $y = c_1e^{2x} + c_2e^{-5x} + \frac{1}{221}(9 \sin 3x - 19 \cos 3x)$.

16. $y = c_1e^{x} + c_2e^{-4x} - \frac{3}{2}$.

17. $y = c_1 + c_2e^{-3x} + \frac{1}{12}x^4 + \frac{1}{3}x^3 - \frac{1}{3}x^2 + \frac{5}{9}x$.

18. $y = (c_1 + \frac{1}{3}x^3 - x^2 - \frac{2}{5}x)\,e^{2x} + c_2e^{-3x}$.

Page 361

19. $y = (c_1 + c_2x + \frac{1}{2}x^2)\,e^{x} + 4\,e^{2x}$.

20. $y = (c_1 + c_2x)\,e^{-x} + x^3 - 6x^2 + 18x - 25$.

21. $y = [c_1 + c_2x - \log(x - 3)]\,e^{3x}$.

22. $y = (c_1 + c_2x)\,e^{-2x} + \dfrac{e^{2x}}{125}(7 \sin 3x - 24 \cos 3x)$.

23. $y = c_1 \sin 2x + c_2 \cos 2x + \dfrac{e^{2x}}{20}(\sin 2x - 2 \cos 2x)$.

24. $y = c_1 \sin x + c_2 \cos x - \frac{1}{3}\cos 2x + \frac{1}{63}\cos 8x$.

25. $y = c_1 \cos 2x + (c_2 + x) \sin 2x$.

26. $y = c_1 \cos x\sqrt{3} + c_2 \sin x\sqrt{3} + \frac{1}{6}x + \frac{1}{2}x \cos 2x - 2 \sin 2x$.

27. $y = e^{2x}[c_1 \sin 3x + (c_2 - \frac{1}{6}x) \cos 3x]$.

28. $y = 1 + \dfrac{1}{3}e^{x} + e^{-\frac{x}{2}}\left(c_1 \cos \dfrac{x\sqrt{3}}{2} + c_2 \sin \dfrac{x\sqrt{3}}{2}\right)$.

29. $y = e^{-x}(c_1 \cos 2x + c_2 \sin 2x) + (\frac{1}{4}x^2 - \frac{1}{8})\,e^{-x} + \frac{1}{60}\,e^{3x}$.

30. $y = e^{\frac{x}{2}}\left(c_1 \cos \dfrac{x\sqrt{3}}{2} + c_2 \sin \dfrac{x\sqrt{3}}{2}\right) + 3x^2 + 10x + 3$.

31. $y = c_1 + c_2x + c_3e^{3x}$.

32. $y = c_1 + c_2 \cos x + c_3 \sin x$.

33. $y = c_1e^{x} + e^{-\frac{x}{2}}\left(c_2 \cos \dfrac{x\sqrt{3}}{2} + c_3 \sin \dfrac{x\sqrt{3}}{2}\right)$.

34. $y = (c_1 + c_2x) \cos x + (c_3 + c_4x) \sin x$.

35. $y = x^2 + c_1x + c_2 + c_3 \cos x + c_4 \sin x$.

36.
$$y = \frac{1}{4}(e^{x} - e^{-x}) + e^{\frac{x}{\sqrt{2}}}\left(c_1 \cos \frac{x}{\sqrt{2}} + c_2 \sin \frac{x}{\sqrt{2}}\right)$$
$$+ e^{-\frac{x}{\sqrt{2}}}\left(c_3 \cos \frac{x}{\sqrt{2}} + c_4 \sin \frac{x}{\sqrt{2}}\right).$$

37.
$$y = c_1e^{-x} + e^{\frac{x}{2}}\left(c_2 \cos \frac{x\sqrt{3}}{2} + c_3 \sin \frac{x\sqrt{3}}{2}\right)$$
$$+ \tfrac{1}{5}\,e^{x}(2 \sin x - \cos x) + \tfrac{1}{13}\,e^{-x}(2 \sin x + 3 \cos x).$$

38. $y = e^{x\sqrt{3}}(c_1 \cos 2x + c_2 \sin 2x) + e^{-x\sqrt{3}}(c_3 \cos 2x + c_4 \sin 2x)$
$\quad + \frac{3}{49}x^3 - \frac{36}{2401}x - \frac{1}{296}e^{-3x}$.

39. $y = c_1 + c_2x + (\frac{1}{3}x^3 - 2x^2 + 7x + c_3)e^x$.

40. $y = (c_1 + c_2x)\cos 2x + (c_3 + c_4x - \frac{1}{64}x^2)\sin 2x$.

41. $y = c_1 + x + (c_2 + c_3x + \frac{3}{4}x^2)e^{2x}$.

42. $y = c_1e^{-x} + c_2 + c_3x - 3x^2 + x^3 - \frac{1}{4}x^4 + \frac{1}{20}x^5$.

43. $x = ce^{\frac{t}{2}},\ y = ce^{\frac{t}{2}}$.

44. $x = ce^{\frac{5t}{2}} + \frac{2}{5}e^{5t} - 2e^{2t},\ y = ce^{\frac{5t}{2}} - \frac{3}{5}e^{5t} - e^{2t}$.

45. $x = c_1e^{3t} + c_2e^{-2t} + \frac{1}{6}e^t + \frac{1}{26}(5\cos 2t + \sin 2t)$,
$y = -3c_1e^{3t} - \frac{1}{2}c_2e^{-2t} + \frac{1}{6}e^t - \frac{1}{52}(17\cos 2t + 19\sin 2t)$.

46. $x = c_1e^t + c_2e^{4t} + \frac{11}{4},\ y = c_1e^t - 2c_2e^{4t} + \frac{1}{2}$.

47. $x = c_1e^{3t} + c_2e^{4t} + 3,\ y = 2c_1e^{3t} + c_2e^{4t} + 2$.

48. $x = c_1 + c_2e^{2t} + c_3e^{-t} + \frac{11}{4}t - \frac{5}{4}t^2 + \frac{1}{6}t^3$,
$y = 2c_1 + \frac{5}{2} - 2c_2e^{2t} + c_3e^{-t} + \frac{5}{2}t - \frac{1}{2}t^2 + \frac{1}{3}t^3$.

49. $x = c_1 + c_2e^{2t} + c_3e^{-t} + \frac{9}{4}t - \frac{3}{4}t^2 + \frac{1}{2}t^3$,
$y = c_1 - \frac{33}{8} + 3c_3e^{-t} - \frac{3}{2}e^{-2t} + 6t - 3t^2 + t^3$.

50. $x = e^{\frac{at}{\sqrt{2}}}\left(c_1 \cos\frac{at}{\sqrt{2}} + c_2 \sin\frac{at}{\sqrt{2}}\right) + e^{-\frac{at}{\sqrt{2}}}\left(c_3 \cos\frac{at}{\sqrt{2}} + c_4 \sin\frac{at}{\sqrt{2}}\right)$,
$y = e^{\frac{at}{\sqrt{2}}}\left(c_2 \cos\frac{at}{\sqrt{2}} - c_1 \sin\frac{at}{\sqrt{2}}\right) + e^{-\frac{at}{\sqrt{2}}}\left(c_3 \sin\frac{at}{\sqrt{2}} - c_4 \cos\frac{at}{\sqrt{2}}\right)$.

Page 362

51. $y = c_1x + c_2x^2 + \frac{1}{2}x^3$.

52. $y = \frac{1}{x^3}[c_1 \cos(\log x^2) + c_2 \sin(\log x^2)] + \frac{x}{100}(5\log x - 2)$.

53. $y = c_1 \cos(2\log x) + c_2 \sin(2\log x) + \frac{1}{4}(\log x)^2 - \frac{1}{8}$.

54. $y = \frac{1}{8}x^2 + c_1(\log x)^2 + c_2 \log x + c_3$.

55. $y = c_1x + c_2\cos(\log x) + c_3 \sin(\log x) - \frac{1}{4x}$.

56. $y = c_1 + c_2x^2 + \frac{c_3}{x^2} - \frac{3}{8}(\log x)^2$.

57. $y = c_1x^3\left(1 + \frac{2\cdot 3}{1\cdot 7}x + \frac{2^2\cdot 3\cdot 4}{\underline{|2}\cdot 7\cdot 8}x^2 + \cdots + \frac{2^r\cdot 3\cdot 4\cdot 5\cdots(r+2)}{\underline{|r}\cdot 7\cdot 8\cdot 9\cdots(r+6)}x^r + \cdots\right)$
$\quad + \frac{c_2}{x^3}(15 + 18x + 9x^2 + 2x^3)$.

58. $y = c_1(6 - 4x + x^2) + c_2x^4\left(1 - \frac{2}{5}x + \frac{3}{5\cdot 6}x^2 - \frac{4}{5\cdot 6\cdot 7}x^3\right.$
$\quad \left. + \cdots + (-1)^r\frac{r+1}{5\cdot 6\cdot 7\cdots(r+4)}x^r + \cdots\right)$.

59. $y = c_1(35 - 42x + 21x^2 - 4x^3) + \frac{c_2}{x^4}(3 - 14x + 21x^2)$.

60. $$y = \frac{c_1}{x^2}(4-x) + c_2x^3\left(1 - \frac{4}{6}x + \frac{4\cdot 5}{\underline{|2}\cdot 6\cdot 7}x^2 - \frac{4\cdot 5\cdot 6}{\underline{|3}\cdot 6\cdot 7\cdot 8}x^3 + \cdots + (-1)^r\frac{4\cdot 5\cdot 6\cdots(r+3)}{\underline{|r}\cdot 6\cdot 7\cdot 8\cdots(r+5)}x^r + \cdots\right).$$

61. $$y = c_1\left[1 - \frac{1}{\underline{|3}}(nx^3) + \frac{1\cdot 4}{\underline{|6}}(nx^3)^2 - \frac{1\cdot 4\cdot 7}{\underline{|9}}(nx^3)^3 + \cdots + (-1)^r\frac{1\cdot 4\cdot 7\cdots(3r-2)}{\underline{|3r}}(nx^3)^r + \cdots\right]$$
$$+ c_2x\left[1 - \frac{2}{\underline{|4}}(nx^3) + \frac{2\cdot 5}{\underline{|7}}(nx^3)^2 - \frac{2\cdot 5\cdot 8}{\underline{|10}}(nx^3)^3 + \cdots + (-1)^r\frac{2\cdot 5\cdot 8\cdots(3r-1)}{\underline{|3r+1}}(nx^3)^r + \cdots\right].$$

62. $$y = c_1\left[1 + \frac{n}{\underline{|2}}x^2 + \frac{n(n-4)}{\underline{|4}}x^4 + \frac{n(n-4)(n-16)}{\underline{|6}}x^6 + \cdots + \frac{n(n-4)(n-16)\cdots[n-(2r-2)^2]}{\underline{|2r}}x^{2r} + \cdots\right]$$
$$+ c_2x\left[1 + \frac{n-1}{\underline{|3}}x^2 + \frac{(n-1)(n-9)}{\underline{|5}}x^4 + \frac{(n-1)(n-9)(n-25)}{\underline{|7}}x^6 + \cdots + \frac{(n-1)(n-9)(n-25)\cdots[n-(2r-1)^2]}{\underline{|2r+1}}x^{2r} + \cdots\right].$$

63. $s = c_1\cos ht + c_2\sin ht + \dfrac{a}{h^2-k^2}\cos kt.$ (h^2 is the constant ratio.)

$s = c_1\cos kt + c_2\sin kt + \dfrac{at}{2k}\sin kt,$ if $h = k.$

64. $$s = e^{-\frac{lt}{2}}\left(c_1e^{\frac{t\sqrt{l^2-4h^2}}{2}} + c_2e^{-\frac{t\sqrt{l^2-4h^2}}{2}}\right) + \frac{a(h^2-k^2)\cos kt + akl\sin kt}{(h^2-k^2)^2+(lk)^2}, \text{ if } l > 2h;$$

$$s = e^{-\frac{lt}{2}}\left(c_1\cos\frac{t\sqrt{4h^2-l^2}}{2} + c_2\sin\frac{t\sqrt{4h^2-l^2}}{2}\right) + \frac{a(h^2-k^2)\cos kt + akl\sin kt}{(h^2-k^2)^2+(lk)^2}, \text{ if } l < 2h;$$

$$s = (c_1 + c_2t)e^{-ht} + \frac{a(h^2-k^2)\cos kt + 2akh\sin kt}{(h^2-k^2)^2+(2hk)^2}, \text{ if } l = 2h.$$

(h^2 and l are the constant ratios.)

65. $h = k.$

CHAPTER XIX

Page 370

1. $z = \phi_1(y)e^{ax} + \phi_2(y)e^{-ax}.$

2. $z = \dfrac{x^{a+1}}{a+1}\phi_1(y) + \phi_2(y).$

3. $z = \phi_1(y)e^{ax} + \phi_2(y).$

4. $z = \phi_1(y)e^{3x} + \phi_2(y)e^{2x}.$

Page 371

5. $z = e^{-2y}[\phi_1(x)\cos y + \phi_2(x)\sin y]$.

6. $z = \frac{1}{4}x^2y^2 + \phi_1(y) + \phi_2(x)$.

7. $z = \frac{1}{3}(x^3y + xy^3) + \phi_1(y) + \phi_2(x)$.

8. $\phi(x - y,\ y - z) = 0$.

9. $\phi(x^2 + y^2,\ z) = 0$.

10. $\phi\left(\frac{y}{x},\ \frac{z^2}{x}\right) = 0$.

11. $\phi(x + y + z,\ x^2 + y^2 - z^2) = 0$.

12. $\phi\left(x^2 + y^2,\ z + \tan^{-1}\frac{y}{x}\right) = 0$.

13. $\phi[e^{-3z}(2x - y),\ e^{-9z}(4x + y)] = 0$.

14. $\phi\left(\frac{y}{x},\ \frac{z}{x}\right) = 0$.

15. $\phi\left(xy,\ \frac{y^3}{3} - xyz\right) = 0$.

16. $\phi\left(\frac{y}{x},\ xy - z^2\right) = 0$.

17. $\phi(x^2 - y^2,\ y^2 - z^2) = 0$.

18. $\phi\left(x - z,\ xz - \frac{y^2}{2}\right) = 0$.

21. $u = \frac{4}{\pi}\left(r\sin x + \frac{1}{3}r^3\sin 3x + \frac{1}{5}r^5\sin 5x + \cdots\right)$.

22. $u = \frac{4}{\pi}\left(e^{-a^2t}\sin x + \frac{1}{3}e^{-9a^2t}\sin 3x + \frac{1}{5}e^{-25a^2t}\sin 5x + \cdots\right)$.

INDEX

(The Roman numerals refer to the volume, the Arabic to the page.)

Abscissa, I, 36
Absolute convergence, II, 283
Absolute value of complex number, II, 305
Absolute value of real number, II, 283
Acceleration, I, 202
Adiabatic line, II, 267
Algebraic equations (*see* Equations)
Algebraic functions, classification, I, 43
  graphs, I, 121
  differentiation, I, 178
  implicit, I, 188
  integration, II, 26, 113, 119
Amplitude of complex number, II, 305
Analytic function, II, 311
Angle between normal and focal radii of ellipse, I, 191
  between plane curves, I, 211
  between planes, II, 186
  between space curves, II, 184
  between straight lines in plane, I, 57
  between straight lines in space, II, 183
  between tangent and radius vector, I, 345
  eccentric, of ellipse, I, 304
  vectorial, I, 329
Arc, derivatives in plane polar coördinates, I, 347
  derivatives in plane rectangular coördinates, I, 196
  differential in plane polar coördinates, II, 79
  differential in plane rectangular coördinates, II, 78
Arc, differential in space rectangular coördinates, II, 180
  length defined, I, 195
  length in plane polar coördinates, II, 77
  length in plane rectangular coördinates, II, 75
  length in space rectangular coördinates, II, 179
  limit of ratio to chord, I, 195
Archimedes, spiral of, I, 332
Area, center of gravity by single integration, II, 94, 96
  center of gravity by double integration, II, 246
  derivative in polar coördinates, I, 348
  derivative in rectangular coördinates, I, 204; II, 47
  determination by double integration, II, 239, 240
  determination by line integral, II, 260
  determination in oblique coördinates, II, 66
  determination in polar coördinates, II, 67
  determination in rectangular coordinates, II, 64
  moment of inertia, II, 236
  of any surface, II, 241
  of ellipse, I, 304; II, 65, 66, 261
  of lemniscate, I, 348
  of parabolic segment, I, 216; II, 95

Area of rose of three leaves, II, 69
    of sphere, II, 242
    of surface of revolution, II, 79
    projection of, II, 177
    representing definite integral, II, 41
Argument of complex number, II, 305
Asymptote, defined and illustrated, I, 128
    of hyperbola, I, 145
    of hyperbolic spiral, I, 333
Attraction, by multiple integration, II, 252
    by single integration, II, 86
    of cylinder, II, 252
    of wire, II, 87
Auxiliary circle of ellipse, I, 304
Axes of coördinates (*see* Coördinates)
    of ellipse, I, 141
    of hyperbola, I, 145
Axis of parabola, I, 147
    of revolution, II, 69
    of symmetry, I, 121
    radical, I, 175

Beams, bending moment of, II, 149
Bernouilli's equation, II, 148
Bessel's equation, II, 359
Bessel's functions, II, 360
Binomial theorem, II, 59
Bisection of straight line, I, 39
Boyle-Mariotte's law, I, 43

Cardioid, I, 337
    radius of curvature, I, 362
Cassini, ovals of, I, 338
Catenary, equation and graph, I, 281
    equation derived, II, 143
    property of, II, 152
Center of circle, I, 134
    of conic, I, 238
    of curvature, I, 356
Center of gravity, defined in plane, II, 90
    defined in space, II, 245
    of elliptic segment, II, 96, 247
Center of gravity, of parabolic segment, II, 95
    of plane area by single integration II, 94
    of plane area by double integration, II, 246
    of plane curve, II, 92
    of quarter circumference, II, 93
    of solid, II, 248
    of solid or surface of revolution, II, 96
    of spherical segment, II, 97
Center of pressure, II, 98
    of pressure of circle, II, 98
Change of coördinates (*see* Coördinates)
    of limits in definite integral, II, 49
Chord of contact, I, 248
    supplemental, I, 264
Circle, auxiliary of ellipse, I, 304
    center of gravity of quarter circumference, II, 93
    definition and general equation, I, 134
    involute of, I, 311
    of curvature, I, 354
    parametric equations, I, 303
    polar equation, I, 342
    pressure on, II, 89, 98
    special case of conic, I, 149
    special case of ellipse, I, 142
    tangent to known line, I, 136
    through known point, I, 136
    through three known points, I, 138
    with center on known line, I, 137
Cissoid, I, 151
    polar equation, I, 341
Clairaut's equation, II, 321
Coefficient, differential, II, 7
    of element of a determinant, I, 8
    undetermined, in differential equations, II, 350
    undetermined, in partial fractions, II, 104
Collinear points in a plane, I, 38
    in space, II, 188
Comparison test for convergence, II, 280

Complex numbers, argument, II, 305
  conjugate, I, 32
  defined, I, 31 ; II, 304
  functions of, II, 307
  graphical representation, II, 304
  modulus of, II, 305
  operations with, I, 32 ; II, 305
Complementary function, II, 349
Components of force, I, 34
  of velocity, I, 200
Concavity of plane curve, I, 112
Conchoid, I, 334
Cone, II, 165
Conics, classification, I, 237
  definition, I, 148
  diameter, I, 252
  general equation, I, 229
  in oblique coördinates, I, 244
  limiting cases, I, 234
  polar, I, 247
  polar equation, I, 343
  tangent, I, 246
  through five points, I, 241
  treatment of numerical equations, I, 240
  See also Ellipse, Hyperbola, Parabola
Conjugate axis of hyperbola, I, 145
Conjugate complex numbers, I, 32
Conjugate diameters of conic, I, 258
Conjugate functions, II, 313
Conjugate hyperbolas, I, 262
Conoid, II, 173
Constant, defined, I, 40
  of integration, I, 206 ; II, 12
Contact, chord of, I, 248
  point of, I, 105
Continuity of function of one variable, I, 101
  of function of two variables, II, 200
Convergence, absolute, II, 283
  comparison test, II, 280
  definition, II, 279
  ratio test, II, 281
  region of, II, 285
Coördinates, Cartesian, I, 224
  change from plane rectangular to oblique, I, 224
  change from plane rectangular to polar, I, 341
  change from space rectangular to cylindrical, II, 231
  change from space rectangular to polar, II, 231
  change in multiple integrals, II, 234
  change in partial derivatives, II, 208
  change of direction of plane axes, I, 221
  change of direction of space axes, II, 193
  change of plane origin, I, 217
  change of space origin, II, 193
  cylindrical, II, 231
  oblique in plane, I, 223
  oblique in space, II, 159
  polar in plane, I, 329
  polar in space, II, 231
  rectangular in plane, I, 35
  rectangular in space, II, 158
Curvature, center of, I, 356
  circle of, I, 354
  definition, I, 353
  radius of, in parametric form, I, 360
  radius of, in polar coördinates, I, 361
  radius of, in rectangular coördinates, I, 354
Curve, Cartesian equation of plane curve, I, 44
  center of gravity, II, 92
  degree, I, 166
  direction of space curve, II, 182
  equations of space curve, II, 170
  parametric equations of plane curve, I, 302
  polar equation of plane curve, I, 330
  second-degree curves, I, 229

Curve, slope of plane curve, I, 99
  tangent to plane curve, I, 104
  tangent to space curve, II, 189
  with given slope, I, 207
  See also Arc, Area, Curvature
Curves, family of, II, 316
  intersection of, I, 161
  with common points of intersection, I, 171
Cycloid, I, 305
  radius of curvature, I, 361
  tangent to, I, 315
Cylinders, II, 169
Cylindroid, II, 173

Definite integrals, applications to geometry, II, 64, 236
  applications to mechanics, II, 86, 236
  change of coördinates, II, 234
  change of limits, II, 49
  defined, II, 39
  differentiation, II, 216
  double, II, 222
  element, II, 64
  evaluation, II, 47
  graphical representation, II, 41
  limits of, II, 40
  properties, II, 45
  triple, II, 230
  with infinite integrand, II, 53
  with infinite limits, II, 52
Degree of plane curve, I, 166
Derivative, applications, I, 202, 203
  application to velocity, I, 198
  defined, I, 102
  higher, I, 111, 187
  higher partial, II, 212, 214
  illustrations of, I, 203
  illustration of partial, II, 199
  of $f(x, y)$ when x and y are functions of s and t, II, 206, 214
  of $f(x, y)$ when x and y are functions of t, II, 202, 214
  of polar coördinates with respect to plane arc, I, 347
Derivative of rectangular coördinates with respect to plane arc, I, 196
  of rectangular coördinates with respect to space arc, II, 182
  partial, II, 198
  second, I, 110
  sign of first, I, 106
  sign of second, I, 111
  theorems on, I, 179
  See also Differentiation
Descartes' folium, I, 132
Descartes' rule of signs, I, 87
Determinants, defined, I, 4
  elements, I, 4
  expansion, I, 8
  minors, I, 4
  properties, I, 6
  solution of equations, I, 1, 12–23
Diameters, conjugate, I, 258–262
  of conic, I, 252
  of ellipse, I, 256
  of hyperbola, I, 257
  of parabola, I, 254
Differential, defined for one independent variable, II, 6
  defined for several independent variables, II, 201
  exact, II, 269
  formulas, II, 10
  graphical representation for one independent variable, II, 8
  graphical representation for two independent variables, II, 206
  higher order, II, 10
  of plane arc, II, 78
  of space arc, II, 180
  partial, II, 202
  total, II, 200
  when equal to zero, II, 209
Differential equations, $(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0$, II, 146
  Bernoulli's, II, 148
  Bessel's, II, 359
  Clairaut's, II, 321
  defined, II, 141

Differential equations, first order, existence of solution, II, 316
  first order not of first degree, II, 318
  Laplace's, in plane, II, 366
  Laplace's, in three dimensions, II, 368
  Legendre's, II, 357
  linear, II, 340
  linear, constant coefficients, any order, II, 348
  linear, constant coefficients, first order, II, 341
  linear, constant coefficients, second order, II, 343
  linear, constant coefficients, solution by partial fractions, II, 347
  linear, constant coefficients, solution by undetermined coefficients, II, 350
  linear, variable coefficients, II, 354
  $Mdx + Ndy = 0$, exact, II, 270
  $Mdx + Ndy = 0$, homogeneous, II, 145
  $Mdx + Ndy = 0$, linear, II, 146
  $Mdx + Ndy = 0$, résumé, II, 317
  $Mdx + Ndy = 0$, variables separable, II, 144
  order of, II, 143
  partial, II, 363
  partial, linear of first order, II, 364
  $Pdx + Qdy + Rdz = 0$, integrable case, II, 328
  $Pdx + Qdy + Rdz = 0$, nonintegrable case, II, 335
  problems in geometry, II, 141, 151, 327
  problems in mechanics, II, 142, 149, 153, 346, 367, 370
  second order, special cases, II, 149
  singular solutions, II, 325
  simultaneous of first order, II, 332
  simultaneous, linear, constant coefficients, II, 353
  solution by series, II, 318, 356
Differentiation, collected formulas, II, 10
  defined, I, 102
  general formulas, I, 184
  of algebraic functions, I, 178
  of definite integrals, II, 216
  of exponential functions, I, 284
  of hyperbolic functions, I, 290
  of implicit functions, I, 188; II, 210
  of inverse hyperbolic functions, I, 292
  of inverse trigonometric functions, I, 276
  of logarithmic functions, I, 284
  of polynomials, I, 103
  of trigonometric functions, I, 272
  of u^n, I, 185
  partial, II, 198
  partial, independent of order, II, 213
  successive, I, 187
  use of logarithm, I, 288
  See also Derivative, Differential
Direction, cosines defined, II, 180
  cosines found from two equations of line, II, 186
  of normal to plane, II, 184
  of plane curve in polar coördinates, I, 345
  of plane curve in rectangular coördinates, I, 197
  of space curve, II, 182
  of straight line in space, II, 180
Directrix of circle, I, 149
  of conic, I, 148
  of ellipse, I, 149
  of hyperbola, I, 149
  of parabola, I, 146
Discontinuity defined, I, 101
  examples of, I, 41, 128, 268, 282, 283; II, 292, 294
  finite defined, II, 291
Discriminant defined, I, 117
  of cubic equation, I, 114, 117
  of quadratic equation, I, 73, 117

Discriminant of quadratic equation in two variables, I, 236
Distance between two points in plane, I, 36
  between two points in space, II, 178
  of point from plane, II, 191
  of point from straight line, I, 63
Divergence, II, 279

e, the number, I, 280
Eccentric angle of ellipse, I, 304
Eccentricity of conic, I, 148
Elasticity, I, 204
Element of definite integral, II, 64
  of determinant, I, 4
Eliminant, I, 23
Elimination, I, 1
Ellipse, angle between normal and focal radii, I, 191
  area, I, 304; II, 65, 66, 261
  center of curvature, I, 357
  center of gravity of segment, II, 96, 247
  definition and simplest equation, I, 139
  evolute, I, 358
  expressed by general equation of second degree, I, 231, 235
  length, II, 77
  parametric equations, I, 303
  special case of conic, I, 148
  radius of curvature, I, 355
  referred to conjugate diameters, I, 259
Ellipsoid, equation, II, 165
  volume, II, 73, 249
Energy, kinetic, I, 203
Entropy, II, 272
Envelopes, II, 322
Epicycloid, I, 307
Epitrochoid, I, 309
Equations, complex roots, I, 82
  depression of, I, 79
  differential (*see* Differential equations)
  discriminant, I, 117
Equations, fractional roots, I, 90
  first degree in two variables, I, 52
  first degree in three variables, II, 161
  irrational roots, I, 92
  multiple roots, I, 116
  Newton's method of solution, I, 114
  number of roots, I, 80
  rational roots, I, 89
  second degree in two variables, I, 229
  simultaneous, I, 161
  simultaneous linear, I, 12
  simultaneous linear homogeneous, I, 21
  solution by factoring, I, 77
  sum and product of roots, I, 82
  transcendental, I, 293
  with given roots, I, 79
Evolute, I, 357
  of ellipse, I, 358
Exact differential in three variables, II, 276
  in two variables, II, 269
Expansion, coefficient, I, 204
Exponential equations, I, 295
Exponential function, differentiation, I, 284
  expansion, II, 58, 308
  integrals, II, 25
  of complex number, II, 307
  of real number, I, 279

Factoring of quadratic polynomial, I, 79
  solution of equations by, I, 77
Factors, integrating, II, 271, 328
  list of integrating, II, 273
  of polynomial, I, 81, 83
Families of curves, II, 316
  orthogonal, II, 314, 327
Flow of liquid, II, 259, 266
Focus of conic, I, 148
  of ellipse, I, 139
  of hyperbola, I, 142
  of parabola, I, 146
Folium of Descartes, I, 132

Force, I, 202
  function, II, 266
Forms, indeterminate, II, 295
Fourier's series, II, 290
Fractions, differentiation, I, 182
  integration, II, 113
  partial, II, 104
Functions, analytic, II, 311
  Bessel's, II, 360
  classes, I, 43
  complementary, II, 349
  conjugate, II, 313
  continuous, one variable, I, 101
  continuous, several variables, II, 200
  decreasing, I, 106
  defined, one variable, I, 40
  defined, several variables, II, 158
  defined by equation of second degree, I, 127
  graphical representation, one variable, I, 44
  graphical representation, two variables, II, 159
  implicit, I, 188; II, 210
  increasing, I, 106
  increment, one variable, I, 100
  increment, several variables, II, 200
  involving fractions, I, 128
  mean value, II, 54
  notation, I, 44
  periodic, II, 290
  total differential, II, 200
  See also Algebraic, Exponential, Hyperbolic, Logarithmic, Transcendental, Trigonometric functions

Gas, graph of Boyle-Mariotte's law, I, 43
  illustrating differential, II, 8
  illustrating function, II, 158
  illustrating partial derivative, II, 199
  temperature-pressure-volume surface, II, 168
General solution, linear differential equation, II, 350
Generators, rectilinear, II, 172
Graph, I, 40
Gravity (*see* Center of gravity)

Harmonic division of line, I, 250
Harmonic motion, I, 275
Harmonic property of polars, I, 249
Heat, as line integral, II, 260
  dependent on path, II, 267
  derivatives, II, 269
  entropy, II, 272
Helix, direction, II, 182
  equation, II, 172
  length, II, 180
Homogeneous differential equation, II, 145
Homogeneous equations, I, 21
Homogeneous functions, II, 145
Horner's method referred to, I, 92
Hyperbola, conjugate, I, 262
  definition and simplest equation, I, 142
  equilateral, I, 146
  expressed by general equation of second degree, I, 231, 235
  referred to asymptotes, I, 224
  referred to conjugate diameters, I, 259
  special case of conic, I, 149
Hyperbolic functions, defined, I, 288
  differentiation, I, 290
  differentiation of inverse, I, 292
  expansion, II, 63
  integrals leading to inverse, II, 24
  inverse, I, 291
Hyperbolic spiral, I, 332
Hyperboloid of one sheet, II, 163
  of two sheets, II, 165
Hypocycloid, I, 309
  four-cusped, I, 132

Imaginary numbers (*see* Complex numbers)
Imaginary surfaces, II, 160
Imaginary unit, I, 31

Implicit functions, general discussion, II, 210
  special case, I, 188
Increment of function of one variable, I, 100
  of functions of several variables, II, 200
Indefinite integral, defined, II, 41
  containing $\sqrt[n]{a+bx}$, II, 29, 119
  containing $\sqrt{a+bx+x^2}$, II, 121
  containing $\sqrt{a+bx-x^2}$, II, 122
  containing $\sqrt{a^2-x^2}$, II, 29
  containing $\sqrt{x^2+a^2}$, II, 29
  containing $\sqrt{x^2-a^2}$, II, 29
  fundamental formulas, II, 26
  of $\dfrac{dx}{(Ax+B)\sqrt{ax^2+bx+c}}$, II, 29
  of $\dfrac{dx}{a+b\cos x}$, $\dfrac{dx}{a+b\sin x}$, $\dfrac{dx}{a\cos x+b\sin x}$, II, 129
  of $e^{ax}\sin bx\,dx$, $e^{ax}\cos bx\,dx$, II, 31
  of rational fractions, II, 113
  of $\sec^n x\,dx$, $\csc^n x\,dx$, II, 127
  of $\sin^n x\,dx$, $\cos^n x\,dx$, II, 123
  of $\sin^m x\cos^n x\,dx$, II, 124, 135
  of $\tan^n x\,dx$, $\text{ctn}^n x\,dx$, II, 126
  of $\tan^m x\sec^n x\,dx$, $\text{ctn}^m x\csc^n x\,dx$, II, 128
  of $\sqrt{a^2-x^2}\,dx$, II, 28
  of $\sqrt{a^2+x^2}\,dx$, II, 32
  of u^n, II, 14
  of $x^m(a+bx^n)^p\,dx$, II, 120, 130
  reduction formulas, II, 130, 135
  See also Integration
Indeterminate forms, II, 295–301
Indicator diagram, II, 43, 54
Inertia (*see* Moment of inertia)
Infinitesimal, defined, II, 1
Infinitesimals, fundamental theorems on, II, 4
  order of, II, 1
Infinity, I, 29
Inflection, point of, II, 112, 194
Integral, particular, II, 349
  See also Definite, Indefinite, and Line integrals
Integrand, defined, II, 12
  infinite, II, 53
Integrating factors, II, 273
Integration, by parts, II, 30, 51
  by substitution (*see* Substitution)
  constant of, II, 12
  definition, II, 12
  elementary formulas, II, 26
  fundamental formulas, II, 13
  of rational fractions, II, 113
  of simple differential equations, II, 141
  possibility of, II, 32
  preliminary discussion, I, 205
  special methods of, II, 119
  See also Definite and Indefinite integrals
Intercepts, I, 53
Intersection, curves with common points, I, 171
  number of points, I, 169
  of plane curves, I, 161
Involute, I, 357
  of circle, I, 311
Isolated points, examples, I, 126; II, 292

Kinetic energy, I, 203

Laplace's equation, in plane, II, 366
  in three dimensions, II, 368
Latus rectum, I, 211
Legendre's coefficients, II, 359
Legendre's equation, II, 357
Lemniscate, I, 340
  area, I, 348
  Cartesian equation, I, 341
Length (*see* Arc and Distance)
Limaçon, I, 336
Limit, approached by variable, defined, I, 97
  of ratio of arc to chord, I, 195
Limits of $\dfrac{\sin h}{h}$ and $\dfrac{1-\cos h}{h}$, I, 270
  of $(1+h)^{\frac{1}{h}}$ and $\dfrac{e^h-1}{h}$, I, 283

Limits, theorems relating to, I, 178
Limits of definite integral, change of, II, 49
  defined, II, 40
  infinite, II, 52
Line, adiabatic, II, 267
Line integral, defined, II, 258
  dependent on path, II, 267
  independent of path, II, 263
  related to double integral, II, 261
Line, straight, determined by two points in plane, I, 60
  determined by two points in space, II, 187
  determined by point and direction in plane, I, 58
  determined by point and direction in space, II, 190
  direction cosines, II, 186
  direction in plane, I, 55
  direction in space, II, 180
  equation in plane, I, 50
  equations in space, II, 170
  intersection in plane, I, 61
  normal equation in plane, I, 64
  parallel lines in plane, I, 56
  parallel lines in space, II, 183
  parametric equations in plane, I, 302
  perpendicular lines in plane, I, 57
  perpendicular lines in space, II, 183
  polar equation in plane, I, 342
  satisfying two conditions in plane, I, 58
  tangent to plane curve, I, 104
  tangent to space curve, II, 189
Linear algebraic equations, I, 1
Linear differential equations, II, 146, 340
Linear partial differential equations, II, 364
  See also Equations and Differential equations
Liquid, flow of, II, 259, 266
Locus, I, 45
  problems, I, 316
Logarithm, Naperian, I, 280
Logarithmic function, differentiation of, I, 284
  expansion, II, 59
  of complex number, II, 310
  of real number, I, 279

Maclaurin's series, II, 55, 287
Maxima and minima defined, I, 108
  discussed by Taylor's theorem, II, 60
  for functions of two variables, II, 205
  problems, I, 192, 275
  test by first derivative, I, 108
  test by second derivative, I, 112
Mean value of function, II, 54
Mechanics and physics problems, acceleration, I, 202
  adiabatic lines, II, 267
  attraction of wire, II, 87
  attraction of cylinder, II, 252
  beams, bending moment, II, 149
  Boyle-Mariotte's law, I, 43
  catenary, II, 142
  center of gravity, II, 90, 245
  center of pressure, II, 98
  coefficient of expansion, I, 204
  components of force, I, 34, 35
  components of velocity, I, 200
  damped vibrations, II, 346
  deflection of girder, I, 110
  elasticity, I, 204
  flow of liquid, II, 259, 266
  force, I, 202
  gas (*see* Gas)
  harmonic motion, I, 275
  heat (*see* Heat)
  height of atmosphere, I, 287
  indicator diagram, II, 43
  kinetic energy, I, 203
  lever, I, 192
  mean velocity, II, 54
  moment of inertia, II, 236, 251
  momentum, I, 203
  motion of particle (*see* Motion)
  pendulum, II, 153

Mechanics and physics problems, potential, II, 203
  potential due to ring, II, 370
  pressure, II, 88
  pressure-temperature-volume surface, II, 168
  projectile, I, 314
  refraction of light, I, 192
  saturated steam, I, 42
  temperature in long plate, II, 367
  uniform motion, I, 199
  velocity, I, 198
  work (*see* Work)
Minima (*see* Maxima)
Minors of determinant, I, 4
Modulus of complex number, II, 305
Moment, bending, II, 149
Moment of inertia, defined, II, 236
  any plane area, II, 236
  rectangle, II, 224
  solid, II, 251
  sphere, II, 251
Momentum, I, 203
Motion of particle, falling in resisting medium, I, 292
  harmonic, I, 275
  in circle, I, 314
  in ellipse, I, 314
  path of, I, 313
  projected upwards, I, 203
  uniform, I, 199
  vibrating in resisting medium, II, 346
  with given velocity and acceleration, I, 207

Newton's solution of equations, I, 114
Normal equation, of plane, II, 185
  of straight line, I, 64
Normal, to plane, II, 184
  to plane curve, I, 191
  to straight line, I, 59
  to surface, II, 204, 211
Numbers, classification, I, 28
  complex, I, 31; II, 304
  irrational, I, 28
Numbers, rational, I, 28
  real, I, 29
  See also Complex numbers

Operator, differential, II, 340
  formulas for $\frac{1}{D-a}$, II, 342
Order of differential equation, II, 143
  of infinitesimal, II, 1
Ordinate, I, 36
Orthogonal trajectories, II, 327
Ovals of Cassini, I, 338

Parabola, area of segment, I, 216; II, 95
  center of gravity of segment, II, 95
  cubical, I, 74
  definition and simplest equation, I, 146
  expressed by general equation of second degree, I, 231, 235
  length, II, 77
  path of projectile, I, 314
  referred to diameter and tangent, I, 255
  referred to tangents at end of latus rectum, I, 132
  semicubical, I, 131
  special case of conic, I, 148
Paraboloid, elliptic, II, 162
  hyperbolic, II, 166
  tangent plane, II, 205
Parametric equations, I, 302
  differentiation, I, 315
Partial derivative, II, 198
Partial differential, II, 202
Partial differential equations, II, 363
Partial fractions, II, 104
Particular integral, II, 349
Parts, integration by, II, 30, 51
Pedal curves, I, 319
Pendulum, II, 153
Periodic function, II, 290
Physics (*see* Mechanics)
Plane determined by normal and point, II, 184
  determined by three points, II, 190

Plane, direction of normal, II, 184
  distance of point from, II, 191
  equation, II, 161
  normal equation, II, 185
  parallel planes, II, 183
  perpendicular planes, II, 183
  tangent, II, 204, 211
  through given straight line, II, 192
Polars, I, 247
  reciprocal, I, 251
Pole of straight line, I, 247
  of system of coördinates, I, 329
Polynomial, defined, I, 43
  derivative of, I, 97
  factors, I, 81, 83
  first degree, I, 50
  *n*th degree, I, 74
  second degree, I, 70
  square root of, I, 121
Potential due to ring, II, 370
  rate of change, II, 203
Pressure, II, 88
  center of, II, 98
  on circle, II, 89, 98
Prismoid, II, 75
Prismoidal formula, II, 74, 173
Products, graphs of, I, 83
Projectile, path of, I, 314
Projection in plane, I, 34
  in space, II, 176

Radius of curvature in parametric form, I, 360
  of curvature in polar coördinates, I, 361
  of curvature in rectangular coördinates, I, 354
  vector, I, 329
Rate of change, I, 203
Ratio test for convergence, II, 281
Rationalization of integrals, II, 119
Reduction formulas, algebraic, II, 130
  trigonometric, II, 134
Region of convergence, II, 285
Resultant, I, 23
Revolution, area of surface, II, 79
Revolution, center of gravity of surface or solid, II, 96
  equation of surface, II, 168
  volume of solid, II, 69
Ring-surface, area, II, 80
  volume, II, 71
Roots, complex, I, 82
  fractional, I, 90
  irrational, I, 92, 114
  location, I, 86
  multiple, I, 116
  number of, I, 80
  of unity, II, 307
  rational, I, 89
  relation to factors, I, 78
  sum and product, I, 82
Rose of three leaves, I, 331
  area, II, 69
Ruled surfaces, II, 172

Segments of line, addition of, I, 32
Series, convergent, II, 279
  divergent, II, 279
  Fourier's, II, 290
  geometric, II, 279
  harmonic, II, 280
  Maclaurin's, II, 55, 287
  operations with, II, 59, 286
  power, II, 284
  solution of differential equations by, II, 318, 356
  Taylor's, II, 55, 287
Sine, graphical construction, I, 267
Singular points of curve, II, 324
Singular solution of differential equation, II, 325
Slope of plane curve, I, 99
  of straight line, I, 54
Solid (*see* Center of gravity, Moment of inertia, and Volume)
Space coördinates (*see* Coördinates)
Sphere, area, II, 242
  center of gravity of segment, II, 97
  equation, II, 178
  moment of inertia, II, 251
Spiral of Archimedes, I, 332

Spiral, hyperbolic, I, 332
  logarithmic, I, 333
Statistics, graphs, I, 41
Stokes's theorem, II, 274
Straight line (*see* Line)
Strophoid, I, 152
Subnormal, I, 210
  polar, I, 351
Substitution, integration by, II, 27
  general discussion, II, 119
  in definite integrals, II, 49
  suggestions for, II, 29
Subtangent, I, 210
  polar, I, 351
Successive differentiation, I, 187
Supplemental chords, I, 264
Surface, normal line, II, 204, 211
  of second order, II, 162–168
  of revolution, II, 168
  ruled, II, 172
  tangent plane to, II, 204, 211
  See also Area, Center of gravity, Moment of inertia, Volume
Sylvester's method of elimination, I, 24
Symmetry, axis of, I, 121

Tangent to circle, I, 190
  to conics, simplest equations, I, 190
  to conics, general equation, I, 246
  to plane curve, I, 104; II, 210
  to space curves, II, 189, 212
  to surface, II, 204, 211
  with given slope, I, 163
Taylor's series for one variable, II, 55, 287
  for several variables, II, 288
Temperature of long plate, II, 367
Time as parameter, I, 313
Total differential, II, 200
Tractrix, II, 142
Trajectories, orthogonal, II, 327
Transcendental equations, I, 293
Transcendental functions of complex number, II, 307–311
Transcendental functions of real number, I, 266
Transformation of coördinates (*see* Coördinates)
Trigonometric equations, I, 293
Trigonometric functions, differentiation, I, 272
  differentiation of inverse, I, 276
  expansion, II, 58, 62, 308
  integrals leading to inverse, II, 19
  integration, II, 123
  inverse, I, 269
  inverse, expansion, II, 59, 62
  of complex number, II, 307
  of real number, I, 266
Trochoid, I, 306

Undetermined coefficients (*see* Coefficients)

Value (*see* Absolute and Mean value)
Variable, defined, I, 40
Variation of sign in polynomial, I, 87
Vector, II, 304
  radius, I, 329
Vectorial angle, I, 329
Velocity, I, 198
  components, I, 200
Vertex of ellipse, I, 141
  of hyperbola, I, 144
  of parabola, I, 147
Volume, any solid, II, 249
  common to two cylinders, II, 74
  cylindrical coördinates, II, 232
  ellipsoid, II, 73, 249
  polar coördinates, II, 233
  prismoidal formula, II, 74
  solid of revolution, II, 69
  solid with parallel bases, II, 72

Witch, I, 149
  points of inflection, I, 194
Work, defined, II, 40
  dependent on path, II, 267
  expressed by indicator diagram, II, 43
  expressed as line integral, II, 259
  independent of path, II, 266

Zero, I, 29

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